Quantum mechanics in terms of discrete beables

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(Received 15 April 1993)

An interpretation of quantum mechanics in terms of classical concepts, "beables," due to de Broglie, Bohm, and Bell (BBB) is generalized and further developed. By assuming that all physical quantities take discrete values on sufficiently small scales, we can use this interpretation to give trajectories for all possible quantities, including the position of a particle, its spin, etc. When applied to position, it is shown that, in the continuum limit, this interpretation reduces to the causal one of Bohm. As an illustration, the BBB trajectories are computed explicitly in two simple models.

PACS number(s): 03.65.Bz

I. INTRODUCTION

In the usual interpretation of quantum mechanics, a description of the subatomic world in terms of classical concepts is rejected. Instead of attributing specific properties to microscopic systems, only the possibility to *observe* these properties is considered. To be specific, we may consider a particle in one dimension. Classically, this particle has a position $x(t)$ and a momentum $p(t)$ for all times t. In a quantum-mechanical description one can still observe that the particle has a position $x(t_1)$ by doing a suitable experiment or one can observe a momentum $p(t_2)$ by doing a different experiment. However, these two experiments are noncompatible (hence, $t_1 \neq t_2$) and the classical picture that the particle at the same time has a sharp position and a sharp momentum cannot be verified. In the Copenhagen interpretation of quantum mechanics, this noncommensurability of noncommuting observables is taken one step further by stating that the quantum particle only *potentially* has a position and momentum; these properties do not actually exist, unless a specific measurement is performed. The belief that it is impossible to assign precise values ("hidden variables") to observables has resulted in proposals for no-go theorems against such hidden-variable interpretations of quantum mechanics [1–4].

It is not clear, however, that the denial of position and momentum on a microscopic level is really unavoidable. In fact, there are attempts to reconcile quantum mechanics with a description in terms of classical concepts such as position and momentum, and Bell has discussed the loopholes in the various no-go theorems in Refs. [2,5]. A realistic interpretation in terms of "*be*-ables" [6] rather than "observables" is appealing because it avoids the notion of a conscious observer and avoids a "cut" between the microscopic, quantum world and our macroscopic world of classical phenomena. Also, it allows one to make sense out of a wave function of the Universe [7,8], which might be relevant for a better understanding of quantum cosmology.

Two of the better known beable interpretations of quantum mechanics are the "causal interpretation" due mainly to Bohm [9,10] (see Ref. [11] for a review) and the "stochastic interpretation" associated with Nelson [12–14] (see Ref. [15] for a review). Both approaches provide trajectories $x(t)$ for the particle position which correspond to a given wave function $\psi(x,t)$. This wave function is a solution of the Schrödinger equation

$$i\hbar\partial_t\psi(x,t) = -\hbar^2\partial_x^2\psi(x,t)/2M + V(x)\psi(x,t) , \tag{1}$$

where M is the particle mass and $V(x)$ the potential.

In the causal approach, the trajectories $x(t)$ are found from ψ by solving

$$\dot{x} = \partial_x S(x,t)/M , \tag{2}$$

with $S(x,t)/\hbar$ the (real) phase of the wave function,

$$\psi(x,t) = R(x,t)\exp[iS(x,t)/\hbar] .$$

In the stochastic approach one has to solve the Langevin equation

$$dx(t) = [\nu\partial_x R(x,t)/R(x,t) + \partial_x S(x,t)/M]dt + \nu^{1/2}d\eta(t) . \tag{3}$$

Here, $\eta(t)$ is white noise with $\langle d\eta(t)\rangle = 0$ and $\langle d\eta(t)^2\rangle = dt$ and ν is an arbitrary, but sufficiently small, diffusion constant $\propto\hbar$. Note that for $\nu = 0$, this equation reduces to the causal one of Eq. (2).

The probability density of an ensemble of particles which move according to Eq. (2) or Eq. (3) obeys the continuity equation

$$\partial_t R^2 = \partial_x(R^2\partial_x S) , \tag{4}$$

which follows from (the imaginary part of) the Schrödinger equation. Assuming that the probability distribution of the particle positions equals $|\psi|^2$ at some initial time t_0, it will be equal to $|\psi|^2$ for all later times as well. This assumption on the initial particle distribution was modified by Bohm [16] which may lead to observable deviations from standard quantum mechanics. In this paper we shall not follow this modification but assume that the initial positions are also distributed according to $|\psi|^2$. Then all predictions of quantum mechanics involving only position are reproduced in terms of particles moving according to Eq. (2) or Eq. (3).

Moreover, one can make contact with the classical equation of motion for the particle as follows: In the stochastic approach one first introduces suitable forward and backward time derivatives D_t^+ and D_t^- (see [14,15] for the definitions). Then one can show that [17]

$$\tfrac{1}{2}M(D_t^+D_t^- + D_t^-D_t^+)x = -\partial_x\left[V - \frac{\partial_x^2 R}{2MR}(\hbar^2 - 4M^2v^2)\right] . \quad (5)$$

Nelson chooses $v=\hbar/2M$ [14] and then this equation resembles Newton's equation, up to the quantum modification $\ddot{x}\to\frac{1}{2}(D_t^+D_t^- + D_t^-D_t^+)x$. For $v=0$, which is the causal case, $D_t^+ = D_t^- = \partial_t$, and one recovers Newton's equation up to a modification of the potential $V\to V+Q$, where the additional piece is the "quantum potential"

$$Q = -\hbar^2\partial_x^2 R/2MR . \quad (6)$$

These approaches show that it is possible to give an interpretation of quantum mechanics where the particles actually have a definite position whether they are observed or not. They give a counterexample to the assertion of the usual Copenhagen interpretation that well-defined and sharp particle trajectories are not allowed by quantum mechanics. It has been objected, however, that these interpretations are too limited: It is not obvious how to extend them to observables that cannot follow continuous trajectories, like spin or fermion number. A related criticism is that position space is singled out and it is not clear why this should prevail over, e.g., momentum space [18,19]. Also, the possibility to generalize these interpretations to relativistic quantum mechanics or field theory has been questioned. These criticisms have already been put forward a long time ago [20–22] and have been answered already to some extent by Bohm [23].

In this paper we focus on yet another beable interpretation, which is less known and less developed than the causal and stochastic interpretations but which appears to obviate the criticisms just mentioned. It will be seen to include the causal interpretation as a special case but it can also be applied to discrete variables like spin and to other nonposition variables like (angular) momentum. Also, an extension to relativistic field theory is, in principle, straightforward.

In Ref. [24] Bell has sketched this beable interpretation which provides stochastic trajectories for discrete quantities (he considers fermion number). Below we shall extend and further develop this approach and we shall illustrate it with explicit examples. We argue that this "de Broglie–Bohm–Bell" (BBB) approach can be applied to any observable, position, momentum, spin, etc., by making the assumption that, on a sufficiently small scale, all quantities take discrete values. It is shown that the position trajectories $x(t)$, computed in the BBB fashion, reduce in the continuum limit (where the discretization can be ignored) to those of the causal approach, or—choosing a different version of BBB—to the stochastic approach.

We shall also argue that this beable interpretation can be taken to an extreme (but in our opinion natural) form where *all* conceivable observables simultaneously have a sharp value, as is the case in a classical description. From a given wave function, one can compute trajectories for any observable O^i, i.e., definite values $v(O^i)$ evolving in time. Of course, these trajectories in general lack the classical correlations, i.e.,

$$v(f(O^1,O^2)) \neq f(v(O^1),v(O^2)) ,$$

where f denotes some arbitrary functional relation. This is obvious for noncommuting operators, but in order to circumvent the Kochen-Specker (KS)-type no-go theorems [3], it must also be the case if the operators commute. Only when ψ is an eigenstate of the commuting operators do the classical correlations hold for these observables on the microscopic level of the trajectories; otherwise, the correlations are only restored in the classical regime.

In the following we shall first explain and generalize the BBB interpretation of quantum mechanics in Sec. II and discuss the relevance of the KS theorem for our approach. In Sec. III we show the connection with the causal and stochastic approaches. A technical part of the argument is deferred to an Appendix. In Sec. IV we illustrate the BBB approach by computing the trajectories for a particle moving on a one-dimensional lattice and for a spinning particle in a magnetic field. Section V contains a discussion of possible objections.

II. DE BROGLIE–BOHM–BELL INTERPRETATION OF QUANTUM MECHANICS

It is sometimes argued [19,24] that in the end all measurements amount to observing (pointer) positions and that it is sufficient to give a beable interpretation for location. Arguing this way, Bell focused on the spatial distribution of fermion number and Bohm exclusively works in the position representation. However, specifying that only a single quantity is a beable with all others being merely observable seems arbitrary, even if this quantity is as fundamental as position. Therefore, we propose to give *all* observables a beable status and we shall explore this point of view below. First, it is shown how Bell's interpretation can be applied to an arbitrary observable. After that, we discuss how the KS theorem, which appears to forbid a realistic interpretation for all observables simultaneously, is circumvented.

A. Trajectories for any observable

In order then to apply Bell's beable interpretation [24] to an arbitrary observable, we shall make the assumption that all physical quantities are discrete and bounded. For those quantities that we observe to be continuous, such as, e.g., the position of a particle, we assume that the minimal separation between two consecutive values is very small, e.g., of the order of the Planck scale. The size of the system is also taken to be finite, such that momenta are also discrete. We shall assume that the dynamics of this finite (but huge) number of degrees of freedom follows from a quantum-mechanical wave function which is

a solution of the (continuous-time) Schrödinger equation.

Consider a solution $|\psi(t)\rangle$ of the Schrödinger equation for a subsystem of this discrete and finite world,

$$i\hbar\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \ , \tag{7}$$

where H is the Hamiltonian for this subsystem. Suppose we want to find the trajectories for an arbitrary but maximal set of commuting observables O^i $(i=1,\ldots,I)$ which have eigenstates $|o^1_{n_1}, o^2_{n_2}, \ldots, o^I_{n_I}\rangle$, with $n^i=1,\ldots,N^i$ labeling the finite and discrete eigenvalues of O^i. For ease of notation we suppress the index i such that the completeness relation simply reads $1=\sum_n |o_n\rangle\langle o_n|$. However, it should be kept in mind that each o_n represents a maximal set of quantum numbers.

The continuity equation in the O representation, which follows from the Schrödinger equation (7), is

$$\hbar\partial_t P_n = \sum_m J_{nm} \ . \tag{8}$$

The probability density P_n and source matrix J_{mn} are defined by

$$\begin{aligned} P_n(t) &= |\langle o_n|\psi(t)\rangle|^2 \ , \\ J_{nm}(t) &= 2\,\mathrm{Im}\{\langle\psi(t)|o_n\rangle\langle o_n|H|o_m\rangle\langle o_m|\psi(t)\rangle\} \ . \end{aligned} \tag{9}$$

In passing we note that we have deviated slightly from Bell's approach, because we assume that the states $\{|o_n\rangle\}$ form a basis in Hilbert space. In Bell's paper [24] the probability density P and source J for a specific observable O^i are considered, while summing over the remaining quantum numbers. In his approach the probability density for observable O^i is then defined by $P^i_n=\sum_q |\langle o^i_n, q|\psi\rangle|^2$, with q denoting the $o^j_{m_j}$, $j\neq i$, and a similar modification in his definition of J^i_{nm}.

We want to interpret this system in terms of beables: At each time the observable actually *has* a value o_n. Since a trajectory for a discrete quantity cannot be continuous (unless it is time independent), we cannot use a differential equation of motion and it is natural to expect stochastic dynamics. In such a description, the jumps of the beable are governed by a transition probability $T_{mn}dt$ which gives the probability to go from state o_n to o_m in the time interval dt. The transition matrix T gives rise to a time-dependent probability distribution of o_n values, $P_n(t)$, which has to satisfy the master equation

$$\partial_t P_n = \sum_m (T_{nm}P_m - T_{mn}P_n) \ . \tag{10}$$

In order to use this stochastic description for the quantum-mechanical system, we have to reconcile (8) with (10). For that, it is sufficient to solve T, for given P and J, from

$$J_{nm}/\hbar = T_{nm}P_m - T_{mn}P_n \ , \tag{11}$$

with $T_{mn}\geq 0$. Since $J_{mn}=-J_{nm}$, this gives only $N(N-1)/2$ equations for the N^2 elements of T and there is a lot of freedom to find solutions. Bell chooses a particular solution where for $n\neq m$,

$$T_{nm} = \begin{cases} J_{nm}/\hbar P_m \ , & J_{nm}\geq 0 \\ 0 \ , & J_{nm}\leq 0 \ . \end{cases} \tag{12}$$

For a given T and an arbitrary, but sufficiently small, time discretization step dt, one can compute trajectories from the transition probabilities $T_{nm}dt$. The probability $T_{nn}dt$ to stay in the same state is not fixed by Eq. (11) or Eq. (12) but follows from the normalization

$$\sum_m T_{mn}dt = 1 \ . \tag{13}$$

However, one can add to the T_{nm} defined in (12) any solution T^0 of the homogeneous equation,

$$T^0_{nm}P_m - T^0_{mn}P_n = 0 \ . \tag{14}$$

Again, there is lots of freedom, but perhaps a natural solution is to take a Gaussian ansatz, $T^0_{nm} \propto \exp(-F^2_{nm}/4\sigma)$, where the antisymmetric part of F_{nm} is chosen to be $m-n$. The symmetric part of F is then fixed by (14), leading to

$$T^0_{nm} \propto \exp\left[-\left[n-m-\frac{2\sigma\ln(P_n/P_m)}{(n-m)} \right]^2 \Big/ 4\sigma \right] \ . \tag{15}$$

For $\sigma\to 0$, this extension of T_{nm} is $\propto\delta_{nm}$ and has no effect on the trajectories, for finite σ, it adds to the probability for jumps over larger values of $|n-m|$.

For the sake of illustration, we write down a third solution,

$$T^0_{nm}=1 \ , \quad T^0_{mn}=P_m/P_n \ , \quad n>m \tag{16}$$

and T^0_{nn} following again from (13).

The choice Bell made looks simpler than extensions with nonzero T^0, but *a priori* any solution of (11) leads to the same time dependence of expectation values involving the observables O^i and is compatible with quantum mechanics. In order to put a constraint on the choices for T_{nm}, we need to judge the resulting trajectories. In particular, we have to verify that the trajectories become solutions of the classical equations of motion in the classical region where $S\gg\hbar$. To investigate this, it is appropriate to use "position" as observable, for which we then must recover Newton's equations of motion.

We shall not carry out this investigation for general solutions of (11) but in Sec. III we shall show that Bell's choice (12) leads to the de Broglie–Bohm "causal" interpretation in the continuous positions limit, and the Gaussian extension (15) leads to Nelson's "stochastic" interpretation, provided we choose the width of the Gaussian sufficiently small. The third solution (16) leads to discontinuous trajectories and must be rejected.

B. Trajectories for all observables

In the above it is shown that a trajectory can be assigned to an arbitrary observable. Given this possibility, one has two options for a realistic interpretation of quantum mechanics: Either one selects a preferred set of observables for which definite values are assumed to exist, or one attempts to assign definite values to all observables

simultaneously. The first option is taken, for example, in the causal and stochastic interpretations. The second option, which we want to advocate in this paper, at first sight seems to be forbidden by the KS theorem.

The KS theorem, in the simplified form discussed in Ref. [4], starts from the assumption that a functional relation among mutually commuting operators should also hold for the values one would assign to them. Denoting the value of an operator O^i by $v(O^i)$, the assumption is that for commuting operators O^i, $i=1,\ldots,n$,

$$v(f(O^1,\ldots,O^n))=f(v(O^1),\ldots,v(O^n)) , \tag{17}$$

where the system may be described by an arbitrary wave function and where f denotes any functional relation. The KS theorem follows from the observation that for certain sets of operators one can write down a number of relations among subsets of mutually commuting operators, which can be shown to violate assumption (17). Since commuting observables can be measured simultaneously, condition (17) is considered to be necessary to avoid conflicts with the predictions of quantum mechanics, and it is concluded that it is not possible to ascribe values to all observables simultaneously.

However, as was recognized already by Bell in Refs. [2,5], there is a hole in the net. Part of the assumption is that the system may be described by an arbitrary wave function, but in fact the constraint (17) must only hold for a wave function during a measurement. If a measurement on the quantum system is performed, one has to consider the combined wave function, with arguments in the much larger space of degrees of freedom of the original system plus measuring device. For a successful measurement, this combined wave function develops sharp peaks with negligible overlap and interference, around configurations which correlate an eigenstate of the quantum system with a state of the measuring device showing the result of the measurement (see, e.g., Ref. [25]). The BBB trajectory of the system follows one particular peak of the wave function. Since there is negligible overlap and interference from the other peaks, these can be neglected for the evolution of the BBB trajectory, just as if the microscopic state has evolved into a mixture of eigenstates. See also Refs. [28,26] for a discussion of the measurement process in the causal interpretation.

If one accepts the constraint that condition (17) must only hold for wave functions during measurement and for observables that commute with the observable being measured, the KS theorem loses its sting and one can use the BBB approach to assign definite values to all observables simultaneously. Toward that end, one simply applies the prescription given above to any chosen set of observables, some of which may mutually commute. In general, the constraint (17) will fail to hold, even if the operators involved commute. However, it was discussed above that during a measurement the wave function of the quantum system effectively evolves into an eigenstate of the observable being measured, and then the constraint (17) holds among any set of operators that commute with the one being measured.

III. CONNECTION WITH THE CAUSAL AND STOCHASTIC INTERPRETATIONS

To simplify the discussion, we consider a particle in one dimension. Following our assumption that all degrees of freedom must be discrete and finite, we restrict the positions to the sites of a one-dimension lattice, $x=an$, with $n=1,\ldots,N$ and a the lattice distance. The Schrödinger equation for wave functions $\psi(x)$ is a discretized version of Eq. (1):

$$\begin{aligned} i\hbar\partial_t\psi(x,t)&=H\psi(x,t)\\ &=-\frac{\hbar^2}{2Ma^2}[\psi(x+a,t)+\psi(x-a,t)\\ &\quad -2\psi(x,t)]+V(x)\psi(x,t) . \end{aligned} \tag{18}$$

Here we have chosen a simple discretization for the Laplacian,

$$\partial_x^2\delta(x-y)\rightarrow[\delta_{n,m+1}+\delta_{n,m-1}-2\delta_{n,m}]/a^2 , \tag{19}$$

with $n=x/a$, $m=y/a$, and $\delta_{m,n}$ the Kronecker δ. Notice that this discrete Laplacian is Hermitian, which is necessary for a unitary time evolution, i.e., to obtain the continuity equation (8) from the Schrödinger equation. The boundary condition on $\psi(x,t)$ will not be important but, to be specific, we shall impose periodicity.

Following the same steps as above, we define the probability density P_n and the matrix J_{mn} as

$$\begin{aligned} P_n&=[\psi(an)]^*\psi(an) ,\\ J_{mn}&=2\,\mathrm{Im}\{[\psi(am)]^*H_{mn}\psi(an)\} . \end{aligned} \tag{20}$$

Here we label P,J and the Hamiltonian H with the integer lattice sites and suppress the argument t. More explicitly, we find, for J_{mn},

$$\begin{aligned} J_{mn}=-\frac{\hbar^2}{Ma^2}\mathrm{Im}\{&[\psi(an+a)]^*\psi(an)\delta_{n,m-1}\\ &+[\psi(an-a)]^*\psi(an)\delta_{n,m+1}\} . \end{aligned} \tag{21}$$

For small discretization step a, and wave functions that are smooth on this lattice scale, we can write, for the shifted wave functions,

$$\begin{aligned} \psi(x+a)=\psi(x)+a\{&[R(x)]'\exp[i\hbar^{-1}S(x)]\\ &+R(x)i\hbar^{-1}[S(x)]'\\ &\times\exp[i\hbar^{-1}S(x)]+O(a)\} , \end{aligned} \tag{22}$$

where we use the polar decomposition $\psi=R\exp(i\hbar^{-1}S)$ and derivatives defined by

$$[F(x)]'=[F(x+a)-F(x)]/a .$$

To leading order in a, it follows that

$$J_{mn}=\frac{\hbar}{Ma}\{[S(an)]'P_n\delta_{n,m-1}-[S(an)]'P_n\delta_{n,m+1}\} . \tag{23}$$

At this point we have to choose a prescription to find the transition matrix T from J. First, we consider Bell's choice which is specified in Eq. (12). Then we find

$$T_{mn}=\begin{cases}\{[S(an)]'/Ma\}\delta_{n,m-1}\ , & [S(an)]'\geq 0\\ -\{[S(an)]'/Ma\}\delta_{n,m+1}\ , & [S(an)]'\leq 0\ .\end{cases} \tag{24}$$

This shows that the nearest-neighbor interactions in the Hamiltonian lead to transitions only between neighboring sites. For positive $[S(an)]'$ the particle can jump from site n to $n+1$ with probability $|[S(an)]'|dt/Ma$, and for negative $[S(an)]'$, it can jump from n to $n-1$ also with probability $|[S(an)]'|dt/Ma$. Since each jump is over a distance a, the average displacement in a time interval dt is $dx=S(x)'dt/M$, and since $S'\rightarrow\partial_x S$ for $a\rightarrow 0$, this suggests that in the continuum limit, the particles have an average velocity $\dot{x}=\partial_x S/M$, as is the case for the trajectories in the causal prescription given in Eq. (2).

To show that the trajectories for $a\rightarrow 0$ become smooth and therefore identical to those of the causal approach, we also have to verify that the dispersion vanishes in the limit that $a\rightarrow 0$. This is carried out in the Appendix, where it is shown that the dispersion in the position vanishes $\propto a^{1/2}$ for $a\rightarrow 0$. On scales which are large compared to the discretization step a, the causal differential equation (2) provides a good description of the particle trajectories. However, on scales comparable to the lattice distance a, this description breaks down and the dynamics is seen to be discontinuous and stochastic.

As discussed above, we can modify the transition matrix by adding an extra term T^0 of the form (15) to T. This extra term introduces transitions between more distant sites and it may lead to deviations from a smooth causal shape of the trajectories in the continuum limit. Let us assume that the width of the Gaussian is sufficiently small, such that jumps are only likely for which we can approximate

$$(\ln P_n-\ln P_m)/(an-am)$$

by $2[R(an)]'/R(an)$. This also requires R to be sufficiently smooth on the lattice scale, as before. Then the transition probability to go from n to m is given by

$$T_{mn}dt\approx\left[\frac{[S(an)]'}{Ma}\delta_{m,n+1}+\alpha\exp(-\{am-an-\sigma a^2[R(an)]'/2R(an)\}^2/2\sigma a^2)\right]dt\ . \tag{25}$$

Here we added αT^0 to the T of Eq. (24) with a free parameter α and assumed that $S'>0$ for definiteness.

From this expression we find the expected value for the jump,

$$\langle m-n\rangle\approx dt\{[S(an)]'/Ma+\alpha\sigma a[R(an)]'/2R(an)\}\ . \tag{26}$$

Also taking into account the dispersion due to the finite width of the Gaussian distribution provided by T^0, we arrive at a Langevin equation for the time dependence of the particle position $x=an$,

$$\begin{aligned}x(t+dt)&=x(t)+\langle am-an\rangle+\tfrac{1}{2}\langle(am-an-\langle am-an\rangle)^2\rangle^{1/2}d\eta+O(a^2)\\&\approx x(t)+\{[S(x)]'/M+(\alpha\sigma a^2)[R(x)]'/2R(x)\}dt+\tfrac{1}{2}(\alpha\sigma a^2)^{1/2}d\eta+O(a^2)\ .\end{aligned} \tag{27}$$

Here, $d\eta$ gives the (rescaled) Gaussian fluctuations, $\langle d\eta^2\rangle=2\,dt$. This coincides with the stochastic equation (3), with $\alpha\sigma a^2/4=\nu$ equal to the diffusion constant. If $\alpha\sigma a^2$ vanishes when $a\rightarrow 0$, then the causal trajectories are recovered; the value $\alpha\sigma a^2=2\hbar/M$, which is finite for $a\rightarrow 0$ but of the order of $\hbar$, reproduces Nelson's stochastic dynamics, whereas still larger values for the dispersion lead to stochastic behavior on a macroscopic scale and are forbidden. This applies in particular to the extreme choice (16) for T^0, where the particle can jump from a given position to any position with an appreciable value of $R(x)$, even in the continuum limit.

Even though a large class of extensions with nonzero T^0 is possible, it is not clear to us what a natural choice would be other than the most simple option $T^0=0$. In the remainder of this paper we shall always use this minimal prescription in our illustrations.

IV. SOME SIMPLE EXAMPLES

To illustrate how the BBB interpretation works in practice, we shall apply it to two simple quantum toy models. First, we consider a free particle moving on a circle and, second we look at the angular momentum of a heavy particle spinning in a magnetic field.

A. Free particle on a circle

For simplicity we consider a particle moving in one dimension. As discussed above, we discretize this system by restricting the positions to lattice sites. For convenience, we shall impose periodic boundary conditions on the wave functions, which implies that the particle moves on a circle. The main objective of this example is to show explicitly how the stochastic BBB trajectories turn into the causal Bohm trajectories in the limit of vanishing discretization $a\rightarrow 0$, $N\rightarrow\infty$ for fixed volume $L=Na$. Therefore, we have chosen to consider a free particle with $V(x)=0$, for which solutions of the Schrödinger equation in the continuum limit and the associated causal trajectories in the Bohm approach can be found easily. From now on, we shall use units in which $\hbar=1$.

In the continuum limit we can find solutions of the Schrödinger equation (1) with $V=0$ which are of Gauss-

ian form:

$$\psi^{\rm cnt}(x,t)=C(t)\sum_{k=-\infty}^{\infty}\exp[-\tfrac{1}{2}A(t)(x+kL-vt)^2 +iMv(x+kL)]\ . \quad (28)$$

Here, $A(t)=(w^2+it/M)^{-1}$, and we have enforced periodicity by the summation over k. This solution represents a wave packet which moves with velocity v and has an initial width w (v and w are free parameters). The factor $C(t)$ is a normalization constant which is irrelevant for the computation of the causal Bohm trajectories.

To find these trajectories, one has to compute the phase of the wave function $\psi^{\rm cnt}$, compute its gradient, and solve the differential equation (2) for a set of initial positions x_0, which are chosen with probability $\propto|\psi^{\rm cnt}(x,0)|^2$. These steps can be carried out easily with the aid of a computer, and a set of trajectories is shown in Fig. 1 (dotted lines). One recognizes the average velocity

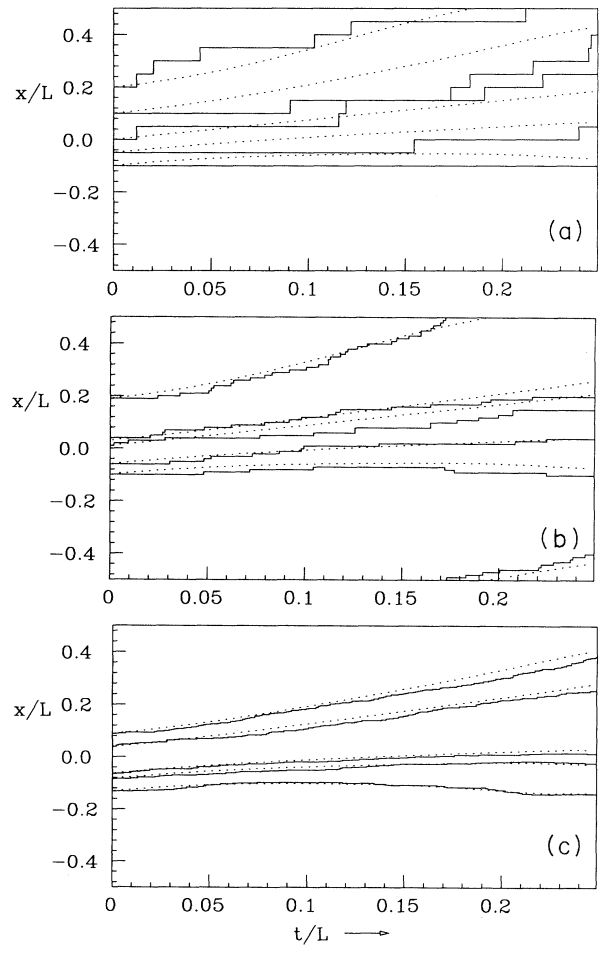

FIG. 1. Trajectories for a free particle. The dotted lines are causal trajectories, computed in the continuum limit; the solid, jumpy lines are discrete BBB trajectories. (a)–(c) have different lattice distance $a=L/N$, with $N=20$, 100, and 500, respectively.

v of the particles, as well as their dispersion due to the spreading of the wave packet.

To find the trajectories in the BBB picture, we follow the steps outlined in Sec. II A. Now the lattice distance is finite and we have to solve the discretized Schrödinger equation (18). To do so, we make a Fourier transformation to momentum space:

$$\tilde{\psi}(p,t)=\sum_x e^{-ixp}\psi(x,t)\ ,$$

$$p=2l\pi/L\ ,\quad l=0,\ldots,N-1\ . \quad (29)$$

In this representation the Hamiltonian is diagonal and has eigenvalues

$$E_p=[1-\cos(ap)]/a^2M\ , \quad (30)$$

which are the lattice equivalents of the familiar $p^2/2M$. For a given initial wave function $\psi_0(x)$, the time-dependent solution is

$$\psi(x,t)=N^{-1}\sum_p\exp[-i(xp+tE_p)]\tilde{\psi}_0(p)\ .$$

In order to compare the discrete model with the version in the continuum limit, we shall use Gaussian initial wave functions as in (28):

$$\psi_0(x)\propto\sum_k\exp[-(x+kL)^2/2w^2+ivM(x+kL)]\ ,$$

with the same width w and initial velocity v.

The (computer) algorithm to find the stochastic trajectories is straightforward: Suppose the particle is at site n; compute the wave function at the neighboring sites and from that the transition matrix $T_{n\pm1,n}$. Since the Hamiltonian couples only nearest neighbors, only transitions from n to $n\pm1$ are possible. Choose a small time step dt to find the probability to jump or to stay at the same site. These probabilities are given by $dt\,T_{n\pm1,n}$ and $1-dt\sum_\pm T_{n\pm1,n}$, respectively. To ensure that this last probability is non-negative, the time step has to be chosen sufficiently small, but is otherwise arbitrary. In practice it is convenient to make the time step temporarily smaller if the particle happens to have jumped to an unlikely position, such that the transition density to move away is very big. Make the decision to jump or stay, using these probabilities, and repeat the whole sequence as many times as desired. To start, choose an initial position $x=na$ with probability $a|\psi_0(x)|^2$.

In Fig. 1 we show a sample of these trajectories for different discretizations. The lattice size $L=Na$ is used to set the scale and the continuum limit is obtained by taking $N\to\infty$ at fixed L. Figures 1(a)–1(c) are for $N=20$, 100, and 500, respectively. In all cases the velocity $v=0.75$, the width of the initial wave packet $w/L=0.15$, and the particle mass $ML=5$. The solid lines are the stochastic paths followed by the particle on the lattice; the dotted lines show the causal paths in the continuum limit which started from the same initial values. The classical trajectories in the continuum would be $x=x_0+vt$, which is approximately valid for the causal trajectories starting near the maximum of the Gaussian wave packet, $x_0=0$. Trajectories which do not coincide

with the path followed by the peak of the wave packet are affected by the quantum potential (6) which drives the particles away from a linear path.

In the stochastic BBB approach, the particle may or may not jump to a neighboring site. The probability of jumping is guided by the wave function, which follows its deterministic evolution. In order for the particle to be able to keep up with this evolution, it must have sufficiently many opportunities to jump. This requires a sufficiently small time step. The paths in Fig. 1 were computed with a time step $dt/L=5\times10^{-5}/N$. The scaling with $1/N$ is necessary to obtain the correct smooth continuum limit (cf. the computation in the Appendix).

On the coarse lattice, $N=20$, the discreteness is clearly visible, and the deviations from the continuum curves are substantial. For increasing N, i.e., smaller lattice distance, it is seen that the particle follows more closely the continuum curves, in accordance with the discussion in Sec. III. This illustrates how the paths which are stochastic on the lattice scale actually resemble smooth and causal trajectories on much larger scales.

Another aspect of the BBB interpretation we wish to emphasize in this paper is the possibility to give complementary (noncommuting) observables simultaneously a beable status. In the above example this means that we can also give trajectories for the momentum of the particle, or for arbitrary combinations of position and momentum like $x+p$. Actually, the momentum trajectories are very simple for a free particle because the Hamiltonian is diagonal in this representation and consequently there cannot be jumps in the value of the momentum, $T_{pq}\propto\delta_{pq}$. Hence, a figure with momentum trajectories for the free particle, whose position trajectories are shown in Fig. 1, would show a set of constant values, distributed according to $|\tilde{\psi}_0(p)|^2$.

Notice that the classical correlation between momentum and position, given by the equality $p=M\dot{x}$, almost never holds exactly. One obvious reason is that on the lattice scale the particle either is at rest or it jumps with infinite velocity. But also when one considers an average velocity, such as in the continuum limit where the causal relation $\dot{x}=S(x)/M$ holds, the classical relation between momentum and change of position in general does not hold. This is clear in the above example, where $\dot{x}$ changes along the trajectories for almost all paths (cf. Fig. 1), but where the momenta are constant. On the average the classical correlation $\dot{x}=p/M$ holds as a consequence of Ehrenfest's theorem. If the approximation that the widths of the wave functions in position and momentum representation can *both* be neglected is valid, the usual classical correlations are approximately recovered for the most probable individual trajectories as well.

B. Spinning particle in a magnetic field

We shall further illustrate the possibility to simultaneously assign trajectories to noncommuting observables using a system for which it is not so obvious that any particular one has a preferred status. We consider a textbook example of a system with discrete quantum numbers, which is provided by a spinning particle in a magnetic field. This example also serves to show that trajectories can be given for truly quantum-mechanical, discrete, quantities like spin, or in this case, angular momentum.

In the approximation that the kinetic energy can be neglected (for large particle mass), the Hamiltonian for the spinning particle is given by

$$H=\mu L_z \ , \tag{31}$$

with the magnetic field chosen in the z direction and μ the magnetic moment. This Hamiltonian is diagonal in the L_z representation,

$$H|m_z\rangle=\mu m_z|m_z\rangle \ , \quad m_z=-l,\ldots,l \ . \tag{32}$$

The integer eigenvalues m_z of L_z range from $-l$ to l, with $l(l+1)$ the eigenvalue of the total angular momentum $L^2=L_x^2+L_y^2+L_z^2$.

With this Hamiltonian the trajectories for L_z are constant in time (the transition matrix for the m_z values is diagonal). Therefore, we focus on the components in orthogonal directions. *A priori* all orthogonal directions are equivalent, but the corresponding components of L do not commute.

To find the trajectories, we need the Hamiltonian and wave functions in an arbitrary representation. It is a straightforward exercise to express L_z in a basis of eigenstates of a linear combination of L_x and L_y:

$$L_\alpha=\cos(\alpha)L_x+\sin(\alpha)L_y \ , \tag{33}$$

which we shall call the α representation. With the aid of a computer, we can then again carry out the BBB prescription to compute trajectories of L_α. As initial wave function we shall choose an eigenstate of L_x with eigenvalue m_x^0. With this choice the initial value of L_x is always $m_x=m_x^0$ without uncertainty, but the initial values for other components L_α, $\alpha\neq0$, are unknown. Only the probability for a particular value is known, $P_{m_\alpha}=|\langle m_\alpha|m_x^0\rangle|^2$.

First, we consider a small value for the total angular momentum, $l=3$. Only for large quantum numbers does one expect classical behavior, and the system for $l=3$ should show strong quantum behavior. This is shown in Fig. 2, where we plot trajectories of L_α for $\alpha=0$, $\pi/4$, and $\pi/2$ [i.e., $L_\alpha=L_x$, $(L_x+L_y)/\sqrt{2}$, and L_y]. The initial value of L_x is $m_x^0=3$. Initial values for other L_α are chosen with probability P_{m_α}. In the point of view taken in this paper, the beable m_α actually follows a single trajectory, one from each figure, and also one for any other value of α not shown here. Of course, it is impossible to predict which particular trajectories will be realized. The discrete nature of L_α is quite pronounced for this small value of l and there is only a vague indication of the constraint $m_\alpha=m_x\cos\alpha+m_y\sin\alpha$ between the trajectories of the various components, which would classically be implied by the relation (33). The same lack of exact correlation would hold for operators which mutually commute (but do not commute with the Hamiltonian). For instance, we could have shown trajectories for the opera-

tors L_α^k, $k=2,3,\ldots$. As discussed in Sec. II B, we cannot impose the KS constraint $v(L_\alpha^k)=v(L_\alpha)^k$, $k=2,3,\ldots$. The trajectories for all L_α^k follow from the same transition matrix T_{mn}, but for each of the L_α^k, they follow from independently chosen transitions. Therefore, it is clear that in general the actual trajectories do not obey the KS constraint. It holds only in the case that the wave function is an eigenstate of L_α.

Next, we consider a larger value for L^2, such that we expect stronger signs of classical behavior. In the classical limit $l\to\infty$, the components of the angular momentum should show Larmor precession with frequency $\omega_{\text{Larmor}}=\mu$. Figure 3 shows some trajectories for a system with $l=20$, again starting from an eigenstate of L_x this time with eigenvalue $m_x^0=16$. Now one begins to see that the trajectories become smoother and are concentrated around two oscillating paths:

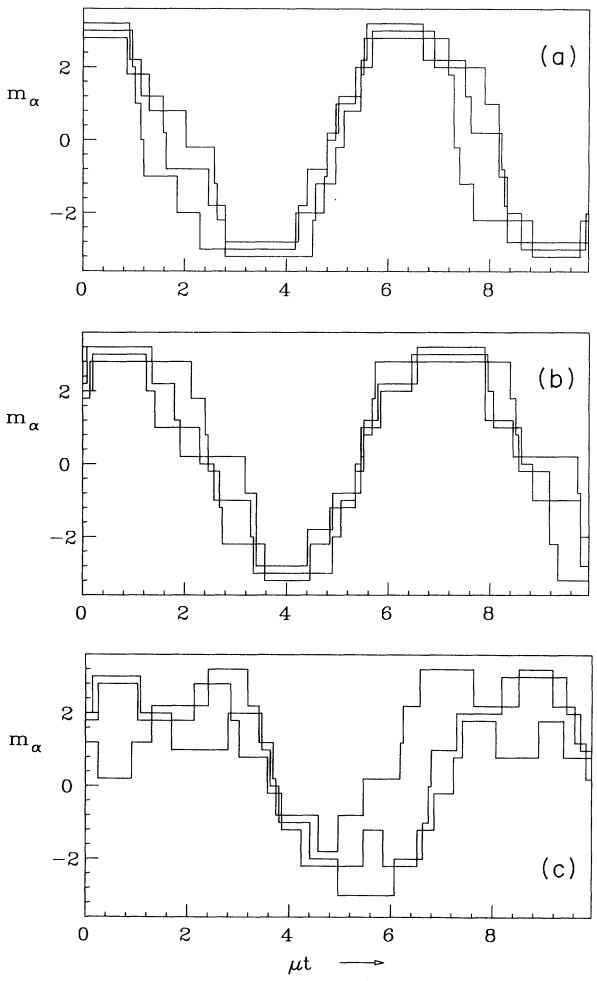

FIG. 2. Three BBB trajectories (for clarity, two of the paths are shifted by ± 0.2) for components of the angular momentum orthogonal to L_z. The total angular momentum $L^2=l(l+1)$ has $l=3$ and time is in units of the magnetic moment μ. (a)–(c) are for $L_\alpha=\cos(\alpha)L_x+\sin(\alpha)L_y$ with $\alpha=0$, $\pi/4$, and $\pi/2$, respectively.

$$m_\alpha \approx l\cos(\mu t\pm\delta-\alpha)\ .$$

The phase shift $\delta=\arccos(m_x^0/l)$, which characterizes the two bunches of trajectories, is such that L_x assumes the initial value $m_x^0=16$ for $t=0$. The period $2\pi/\mu$ is characteristic of Larmor precession.

In particular, it is seen that the classical correlation between the (infinitely many) components L_α is getting restored: For all α, the trajectories are concentrated around paths given by

$$m_\alpha = l\cos(\mu t\pm\delta+\alpha)\ .$$

These paths give the positions of the peaks in the wave function. Of course, it is not surprising to find that quantum-mechanical averages approximate the correct classical behavior, since this is guaranteed by Ehrenfest's theorem. However, it is a virtue of the BBB interpretation that it can show how this comes about in a very explicit way.

It is a nice feature of the BBB dynamics that it allows

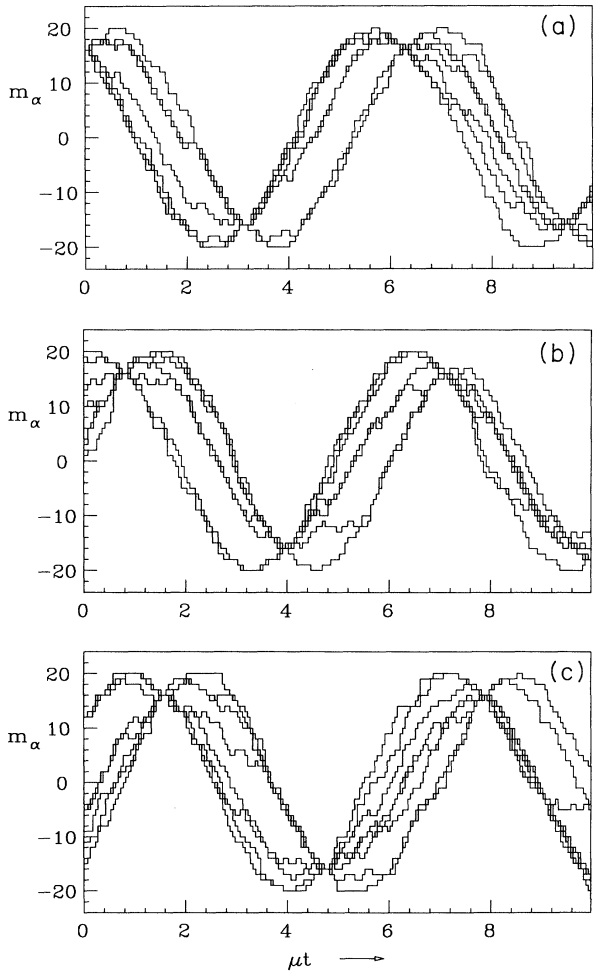

FIG. 3. The BBB trajectories for components of the angular momentum as in Fig. 2, but now for a larger value of the total angular momentum, $l=20$.

for a random bifurcation into two approximate classical trajectories starting from the same initial value. The wave function, e.g, in the L_x representation, is peaked along the two classical trajectories,

$$m_x = l\cos(\mu t \pm \delta) ,$$

where

$$\delta = \arccos(m_x^0/l) \approx 0.2\pi$$

in Fig. 3. The paths followed by these peaks intersect at $\mu t = k\pi$ (integer k). Looking at the distribution of a large number of trajectories, one indeed recognizes this double-peak structure of the wave function in the distribution of the paths, and one would think from Fig. 3 that the individual trajectories are like classical ones which are afflicted with quantum fluctuations. One would then expect that these fluctuations are getting suppressed for larger values of l such that the smooth classical oscillations are recovered for $l \to \infty$.

This is almost true, except that the trajectories can bifurcate when the wave packet guiding the jumps of m_x intersects with the second, empty wave packet. Then they can interchange their roles and the wave packet which previously was empty can become the guiding one. This is illustrated in Fig. 4, where we plot a single trajectory, together with the two classical paths starting from the same initial value as the quantum path. One clearly sees that the particle path first follows one wave packet but switches to the second one at $\mu t \approx 3 \approx \pi$, when the two packets intersect. This unpredictability of the path is a salient feature of the stochastic BBB approach, which is absent in a causal description.

Of course, such a bifurcation should not occur for macroscopic observables. The reason we find it in the above example is that the wave function is very special: It has *two* peaks moving along classical paths, with strong interference and large overlap between these wave packets at $\mu t = k\pi$ when they intersect. This is a consequence of choosing an exact eigenstate of L_x as initial state instead of an appropriate coherent state. When interference effects are strong, the particle trajectory may switch from one packet to the other. For macroscopic wave functions, interference and overlap between different wave packets is strongly suppressed and bifurcation of macroscopic trajectories is extremely improbable.

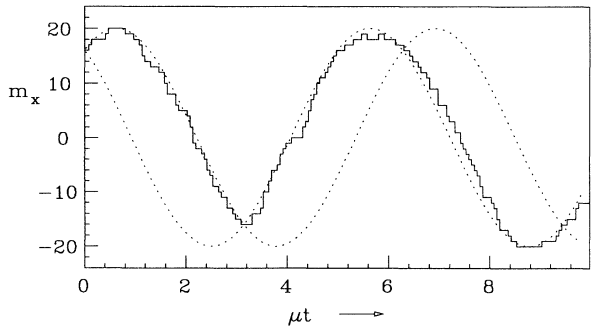

FIG. 4. Bifurcation of the BBB trajectories for L_x. The dotted lines represent the classical paths $m_x = l\cos(\mu t \pm \delta)$, $\delta \approx 0.2\pi$; the solid line is one of the quantum paths from Fig. 3(c).

V. DISCUSSION

In this paper we have investigated a proposal for a realistic interpretation of quantum mechanics which is a generalization of Bell's beable approach sketched in Ref. [24]. Contrary to the expectation based on the usual assertions that quantum mechanics is incompatible with classical concepts, beables, on a microscopic level, it was seen in Sec. II that such beable interpretations are in fact very easy to give. The difficulty is rather that the perspective offered in that section allows for too many beable interpretations which are compatible with quantum mechanics on the level of observable, measurable properties. The only guideline we then have is that the beable dynamics must reduce to classical dynamics in the appropriate limit.

Even then, various versions of a beable interpretation according to BBB are possible. In our preferred version we compute trajectories for a maximal set of commuting observables; Bell only considers a subset. In the causal and stochastic interpretations one only considered positionlike quantities; we propose to assume that trajectories are realized for all possible quantities, with simultaneously sharp values. Finally, there is the freedom to extend the transition matrix T with a nonzero T^0. This freedom is reflected in the known possibility [17] to interpolate between Bohm's causal and Nelson's stochastic interpretations. In our view the most natural strategy is to consider all possible maximal sets of commuting observables, with the minimal choice for the transition matrix, $T^0 = 0$. Of course, these choices are by no means compelling.

A feature which is central to our approach is that on a fundamental level all physical quantities are assumed to be discrete. Starting from quite different considerations stemming from difficulties with black-hole entropy, this point of view was also put forward in Ref. [27]. When all quantities are discrete, one would also expect that time should be discrete. In fact, one can further generalize the discussion of Sec. II A to a form which only assumes (discrete) unitary evolution of the wave functions. However, this introduces further ambiguities, which are only restricted by the requirement that the usual Schrödinger evolution is recovered in the continuous time limit. This seems to make a discrete time approach rather contrived, and we have not pursued this interesting possibility in this paper.

It is also worth stressing that, unlike the causal interpretation, the BBB interpretation is not deterministic. Even though the evolution of the wave functions is strictly deterministic and time reversible, the actual trajectories are stochastic on a fundamental level. This can lead to a strictly unpredictable time evolution. This happens, for instance, if the small stochastic jumps take the particle to a "neighboring" trajectory which deviates exponentially with time from the original one. This is the case, e.g., if a wave packet splits up, or if it intersects with one (or more) other wave packets, leading to large

interference effects. An example of this switching from one wave packet to another was shown in Fig. 4.

It is often suggested that beable interpretations of quantum mechanics are in conflict with (general or special) relativity [11,28,29]. On the one hand, the stochastic evolution of the beables singles out a frame in which the jumps to new values take place simultaneously. This singles out an absolute time and a three-dimensional space, which is the border between "past" space-time regions, where the beables have acquired their actual values, and the "future" with only probabilities. This would obviously be at variance with special relativity. Recently, a detailed argument has been presented in Refs. [30,31] that a conflict between beable trajectories and Lorentz invariance is unavoidable. On the other hand, the BBB approach is sufficiently general to make it also applicable to relativistic field theory in a Hamiltonian, Schrödinger formulation. The fields should then be regularized on a finite lattice, and should take discrete values. Here the underlying dynamics of the wave functionals is relativistically covariant (at least in the continuum limit) and one can speculate that only this relativistic covariance is relevant, whereas it is broken on the level of the actual trajectories which evolve in a preferred frame.

ACKNOWLEDGMENTS

I would like to thank L. Hardy, J. Smit, and E. Squires for various stimulating discussions. This work was supported by the U.S. DOE under Contract No. DE-FG03-91ER40546.

APPENDIX

In this appendix it is shown in some detail that the stochastic trajectories with the minimal choice for the transition density T given in Eq. (12) reduce to the causal trajectories of Bohm in the continuum limit $a \to 0$. We start from the approximate form of the transition matrix given in Eq. (24) above:

$$T_{n+1n} \approx dt[S(an)]'/Ma \ , \tag{A1}$$

which is valid for wave functions which are sufficiently smooth on the lattice scale a. For definiteness, consider a particle with position $x=na$ at $t=0$, for which $[S(x)]'>0$. Then only T_{n+1n} and T_{nn} are nonzero. More precisely, we shall assume that S' is approximately constant on an interval $(x,x+\Delta L)$. This interval is very small on a macroscopic scale, but contains many lattice sites: $N=\Delta L/a$ is big. We shall further assume that S' remains approximately constant in a macroscopically small time interval Δt, which, however, contains many time steps $dt=\Delta t/N$. Notice the scaling of dt with $1/N$ which was mentioned in the main text.

Given these conditions, we can compute the expected value for the position of the particle at $t=\Delta t$ as well as the dispersion. Writing

$$dt\, T_{n+1n} = dt[S(x)]'/Ma = p$$

and $dt\, T_{nn} = (1-p)$ for all $na \in (x,x+\Delta L)$, the expected change in position after the time interval Δt is given by

$$\begin{aligned}\langle ka \rangle &= \sum_{k=0}^{N} ka \begin{bmatrix} N \\ k \end{bmatrix} p^k (1-p)^{N-k} \\ &= ap \frac{d}{dp}(p+q)^N|_{q=1-p} = Nap \ . \end{aligned} \tag{A2}$$

This gives the result stated in the main text, which is that

$$\langle \Delta x \rangle = Na\{dt[S(x)'/Ma\} = \Delta t[S(x)]'/M \ , \tag{A3}$$

which is equal on the average to the velocity of the causal trajectories, $\langle \Delta x \rangle/\Delta t \approx \partial_x S(x)/M$. The $\approx$ sign is a reflection of the assumptions on S' we made in arriving at this result.

Similarly, we can compute the dispersion in the average displacement:

$$\begin{aligned}\mathrm{disp}(\Delta x) &= [\langle (ka)^2 \rangle - \langle ka \rangle^2]^{1/2} \\ &= a[N(1-p)]^{1/2} = a^{1/2}(\Delta L - \langle \Delta x \rangle)^{1/2} \ . \end{aligned} \tag{A4}$$

The continuum limit must be taken such that $a \to 0$ for a fixed macroscopic value of L; therefore, it is seen that the dispersion in Δx indeed vanishes $\propto a^{1/2}$, as claimed in the main text. This completes the proof that the stochastic BBB trajectories (with the minimal choice for T) reduce to the deterministic Bohm trajectories in the continuum limit.

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