when Eq. (B2) is satisfied. Equation (B3) can then be written in the form

$$a^{-2}\gamma(a^2\gamma\beta_\pm')' = -\tfrac{1}{2}\partial V/\partial\beta_\pm . \qquad \text{(B8)}$$

To see if Eq. (B8) is equivalent to the geodesic equation (28), transform the curve parameter from Ω to ω. Constraint (25) yields the equation

$$d\Omega/d\omega = \pm\gamma , \qquad \text{(B9)}$$

which connects the parameters Ω and ω. It is then a straightforward matter to compute $d^2\Omega/d\omega^2$ and show that the first geodesic equation (27) follows from the definition of ω alone. By using Eq. (B9) one can then show that

$$a^{-2}(a^2\dot\beta_\pm)^\cdot = a^{-2}\gamma(a^2\gamma\beta_\pm')'$$

so that Eq. (B8) reduces to the remaining geodesic equations (28).

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Insolubility of the Quantum Measurement Problem*

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The problem of whether a measurement interaction can leave the joint object-apparatus system in a mixture of states, in each state of which the apparatus's observable displays a definite value, is set within the most general quantum-theoretic framework for treating measurements. It is shown that the question posed by this problem admits only a negative answer. Some schemes for approximating the true object-apparatus state by means of such mixtures are examined. It is argued that such schemes constitute fundamental changes in the interpretation of quantum theory.

I. INTRODUCTION

THE quantum theory of measurement pursues the idealization where the measured object, the measuring apparatus, and the interaction between the two are each treated within the formalism of quantum theory. If both object and apparatus have, as measurement begins, a pure state, then, since the interaction between them is represented by a unitary motion on the joint object-apparatus space, the terminal state of the joint object-apparatus system will be a pure case in which, generally, neither the object nor the apparatus has a definite state. If one thinks of the apparatus as a macroscopic device—say, a pointer and scale—then the result that the apparatus has no state function is unacceptable. One may try to avoid this result by treating the initial state of the apparatus (more realistically, one may argue) as a mixed state and then hoping that the final state of the joint system will be a mixture of pure states in each of which the apparatus is itself in a pure state. The question of whether this can successfully be done is known as "the problem of measurement." For measurements satisfying von Neumann's account,[1] Wigner has shown that the problem of measurement cannot be solved affirmatively.[2] D'Espagnat[3] and Earman and Shimony[4] have generalized Wigner's argument for the broader class of measurements that fall under Landau's analysis.[5] I shall outline below the most general theory of measurement consistent with elementary quantum theory, an account which includes as special cases the theories of von Neumann and Landau, and by a somewhat different argument I shall show that no affirmative solution to the problem of measurement is possible. The remaining section will investigate the prospects for an approximate solution.

II. PROBLEM

We shall consider an object system with associated Hilbert space $\mathbf{H}_o$ and an apparatus system with space $\mathbf{H}_a$. An apparatus observable $\mathbf{A}$ with spectral resolution $\mathbf{A}=\sum\mu_n\mathbf{A}_n$ will be used to measure an object observable $\mathbf{O}$ with spectral resolution $\mathbf{O}=\sum\lambda_n\mathbf{O}_n$. The interaction will be treated in the tensor product space $\mathbf{H}=\mathbf{H}_o\otimes\mathbf{H}_a$. For generality, "states" will always be mixed, unless otherwise indicated, and the density operator-trace formalism will be used. Thus if initially the object has state $\mathbf{W}_o$ and the apparatus has state $\mathbf{W}_a$, the joint system will have state $\mathbf{W}_o\otimes\mathbf{W}_a$. The measurement is effected by means of a unitary motion $\mathbf{U}$ on $\mathbf{H}$ so that when the measurement terminates, the joint system has state $\mathbf{U}(\mathbf{W}_o\otimes\mathbf{W}_a)\mathbf{U}^{-1}$.

It is widely believed that there are no measurements that will leave both object and apparatus in definite

* Work supported by NFS under Grant No. GS-2034.
[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, translated by Robert T. Beyer (Princeton U.P., Princeton, 1955), Chaps. 5 and 6.
[2] E. P. Wigner, Am. J. Phys. **31**, 6 (1963). The same result is contained in A. Komar, Phys. Rev. **126**, 135 (1962).
[3] B. d'Espagnat, Nuovo Cimento Suppl. **4**, 828 (1966).
[4] J. Earman and A. Shimony, Nuovo Cimento **54B**, 332 (1968).
[5] L. Landau and R. Peierls, Z. Physik **69**, 56 (1931); L. Landau and E. Lifshitz, *Quantum Mechanics* (Pergamon, London, 1958), pp. 21–24.

pure states (i.e., where the final state of the joint system factors) and that this is the source of the problem of measurement. But suppose that $\mathbf{O}$ and $\mathbf{A}$ have infinite, multiplicity-free spectra. Let $\{\phi_i\}$, $\{\xi_i\}$ be complete orthonormal bases for $\mathbf{H}_o$, $\mathbf{H}_a$ made up of eigenstates of $\mathbf{O}$, $\mathbf{A}$. Then $\{\phi_i \otimes \xi_j\}$ is an orthonormal basis for $\mathbf{H}$ and

$$\mathbf{U}(\phi_i \otimes \xi_j) = \phi_j \otimes \xi_i \tag{1}$$

defines a unitary operator $\mathbf{U}$ on $\mathbf{H}$. If initially the object had pure state $\phi = \sum a_i \phi_i$ and the apparatus had pure state $\xi = \sum b_i \xi_i$, then the terminal state under $\mathbf{U}$ would be

$$\mathbf{U}(\phi \otimes \xi) = (\sum b_i \phi_i) \otimes (\sum a_i \xi_i). \tag{2}$$

Thus the final state of the joint system factors so that both object and apparatus have definite pure states. Clearly interactions corresponding to $\mathbf{U}$ would count as measurements[6] since determining the final apparatus state $\sum a_i \xi_i$ would yield the a_i (or at least the $|a_i|^2$) characterizing the initial, unknown object state ϕ. If initially the apparatus were in the mixed state $\mathbf{W}_a = \sum w_n \mathbf{P}_{[\xi_n]}$, then the final state of the joint system would be

$$\begin{aligned}\mathbf{U}(\mathbf{P}_{[\phi]} \otimes \mathbf{W}_a)\mathbf{U}^{-1} &= (\sum w_n \mathbf{P}_{[\phi_n \otimes \Sigma a_i \xi_i]}) \\ &= (\sum w_n \mathbf{P}_{[\phi_n]}) \otimes \mathbf{P}_{[\Sigma a_i \xi_i]}. \end{aligned} \tag{3}$$

(I use $\mathbf{P}_{[x]}$ for the operator projecting on the subspace spanned by x. By examining x no confusion should arise as to which Hilbert space the operator is defined on.) Here the result of the measurement is to leave the joint system in a mixture of states in each of which both objects and apparatus have definite pure states, indeed, in each of which the apparatus has the same pure state $\sum a_i \xi_i$. But this does not count as a solution to the problem of measurement since the final apparatus state is a superposition of states corresponding to definite pointer positions (the ξ_i). Thus, although $\sum a_i \xi_i$ might, for instance, be an energy eigenstate, it would not correspond to any pointer position whatsoever; i.e., in the superposed state $\sum a_i \xi_i$ the apparatus observable $\mathbf{A}$ takes on no value at all. It is the anomaly of this feature that constitutes the problem of measurement. One can now formulate that problem more precisely as follows: Are there measurements that always leave the state of the joint object-apparatus system in a mixture in each state of which the apparatus observable displays a definite eigenvalue.

For eigenvalue μ of $\mathbf{A}$, let $\mathbf{H}_\mu$ be the subspace of $\mathbf{H}_a$ spanned by the eigenstates of $\mathbf{A}$ belonging to μ. Let $\mathbf{H}_o \otimes \mathbf{H}_\mu$ be the subspace of $\mathbf{H}$ spanned by the tensor product of basis vectors of $\mathbf{H}_o$ and basis vectors of $\mathbf{H}_\mu$. If ψ is in $\mathbf{H}_o \otimes \mathbf{H}_\mu$, then $(\mathbf{I} \otimes \mathbf{A})\psi = \mu\psi$ (where $\mathbf{I}$ is the identity operator on $\mathbf{H}_o$) and we can say that in ψ the apparatus observable $\mathbf{A}$ takes on (or "displays" or "has") the value μ. The problem of measurement, then, is the following question: *Are there unitary operators* $\mathbf{U}$ *on* $\mathbf{H}$ *corresponding to a measurement from the fixed initial apparatus state* $\mathbf{W}_a$ *such that for any initial object state* $\mathbf{W}_o$

$$\mathbf{U}(\mathbf{W}_o \otimes \mathbf{W}_a)\mathbf{U}^{-1} = \sum w_n \mathbf{P}_{[\beta_n]}, \tag{4}$$

where for each n there is some eigenvalue $\mu(n)$ of $\mathbf{A}$ such that β_n belongs to $\mathbf{H}_o \otimes \mathbf{H}_{\mu(n)}$.

Before one can proceed further, criteria must be established for when a unitary operator corresponds to a measurement from a fixed initial apparatus state.

III. THEORY OF MEASUREMENT

The basic requirement[7] for a measurement is that observations of the apparatus observable $\mathbf{A}$ made on the state of the joint system that results from the measurement should yield requisite information about the initial, unknown state of the object with regard to the object observable $\mathbf{O}$. The minimum requirement of this sort is that one should be able to distinguish between states of the object that yield different probabilities for some value of $\mathbf{O}$.

Definition 1. Call states $\mathbf{W}_o$, $\mathbf{W}_o'$ [$\mathbf{W}_a$, $\mathbf{W}_a'$] $\mathbf{O}$-*distinguishable* [$(\mathbf{I} \otimes \mathbf{A})$-*distinguishable*] if for some n,

$$\operatorname{tr}(\mathbf{W}_o'\mathbf{O}_n) \neq \operatorname{tr}(\mathbf{W}_o\mathbf{O}_n)$$
$$[\operatorname{tr}(\mathbf{W}_a'(\mathbf{I} \otimes \mathbf{A}_n)) \neq \operatorname{tr}(\mathbf{W}_a(\mathbf{I} \otimes \mathbf{A}_n))];$$

otherwise $\mathbf{O}$-*indistinguishable* [$(\mathbf{I} \otimes \mathbf{A})$-*indistinguishable*].

Then we can make the following definition of measurement interactions.

Definition 2. A unitary operator $\mathbf{U}$ is a $\mathbf{W}_a$ *measurement* (of the observable $\mathbf{O}$ by means of the observable $\mathbf{A}$) if and only if whenever $\mathbf{W}_o$, $\mathbf{W}_o'$ are $\mathbf{O}$-distinguishable, then $\mathbf{U}(\mathbf{W}_o \otimes \mathbf{W}_a)\mathbf{U}^{-1}$ and $\mathbf{U}(\mathbf{W}_o' \otimes \mathbf{W}_a)\mathbf{U}^{-1}$ are $(\mathbf{I} \otimes \mathbf{A})$-distinguishable.

The requirement embodied in Def. 2 is, of course, a minimal one and in no way ensures that all the requisite information about the object can be obtained by a $\mathbf{W}_a$ measurement. One can achieve a more complete description as follows.

Definition 3. A $\mathbf{W}_a$ measurement $\mathbf{U}$ is a $\mathbf{W}_a$ *filter* if and only if whenever $\mathbf{W}_o$, $\mathbf{W}_o'$ are $\mathbf{O}$-indistinguishable then $\mathbf{U}(\mathbf{W}_o \otimes \mathbf{W}_a)\mathbf{U}^{-1}$ and $\mathbf{U}(\mathbf{W}_o' \otimes \mathbf{W}_a)\mathbf{U}^{-1}$ are $(\mathbf{I} \otimes \mathbf{A})$-indistinguishable.

Let

$$b_n = \operatorname{tr}[\mathbf{U}(\mathbf{W}_o \otimes \mathbf{W}_a)\mathbf{U}^{-1}(\mathbf{I} \otimes \mathbf{A}_n)]$$
$$[= \text{the probability for } \mu_n \text{ in the final state } \mathbf{U}(\mathbf{W}_o \otimes \mathbf{W}_a)\mathbf{U}^{-1}]. \tag{5}$$

Where ψ_k belongs to λ_k, let

$$a_{nk} = \operatorname{tr}[\mathbf{U}(\mathbf{P}_{[\psi_k]} \otimes \mathbf{W}_a)\mathbf{U}^{-1}(\mathbf{I} \otimes \mathbf{A}_n)] \tag{6}$$

[6] J. Albertson, Phys. Rev. 129, 940 (1963), develops an account of measurement using these interactions. These interactions, of course, are not of the tradiational sort associated with the theory of von Neumann. See Ref. 7.

[7] Details of the theory sketched below can be found in A. Fine, Proc. Cambridge Phil. Soc. 65, 111 (1969).

(=the probability for μ_n as the result of a $\mathbf{W}_a$ measurement initiated in object state ψ_k).

Let

$$x_k = \mathrm{tr}(\mathbf{W}_o\mathbf{O}_k) \tag{7}$$
(=the probability for λ_k in state $\mathbf{W}_o$).

Clearly the b_n can be determined by observations on the apparatus from state $\mathbf{U}(\mathbf{W}_o\otimes\mathbf{W}_a)\mathbf{U}^{-1}$. The a_{nk} can similarly be determined by preparing initial eigenstates ψ_k. The x_k constitute precisely the desired information characterizing the initial, unknown object state $\mathbf{W}_o$ with regard to the object observable $\mathbf{O}$. The requirement that $\mathbf{U}$ be a $\mathbf{W}_a$ filter guarantees that

$$b_n = \sum_k a_{nk}x_k\,. \tag{8}$$

Thus measurements by means of a filter reduce to solving the system (8) of linear equations (for $n=1, 2, \ldots$) for the unknowns x_k in terms of the known b_n, a_{nk}. The uniqueness of the solution is ensured.[8]

So far as I can tell, the above notion of a filter captures the most general possible concept of quantum measurement. The more specific proposals of Refs. 1, 5, and even 6 fall under this general notion.

IV. INSOLUBILITY OF PROBLEM OF MEASUREMENT

One can now show with complete generality that the answer to the question posed by the problem of measurement is in the negative: There are no $\mathbf{W}_a$ measurements $\mathbf{U}$ such that (4) holds for all initial object states $\mathbf{W}_o$.

To show this, suppose that ϕ_1, ϕ_2 are eigenstates of $\mathbf{O}$ belonging to distinct eigenvalues and suppose that $\mathbf{W}_a = \sum w_n \mathbf{P}_{[\gamma_n]}$. Then if $\mathbf{F}_i = \mathbf{U}(\mathbf{P}_{[\phi_i]}\otimes\mathbf{W}_a)\mathbf{U}^{-1}$, we have

$$\mathbf{F}_i = \sum_n w_n \mathbf{P}_{[\beta_{in}]}\,, \tag{9}$$

where $\beta_{in} = \mathbf{U}(\phi_i\otimes\gamma_n) \in \mathbf{H}_o\otimes\mathbf{H}_{\mu^i(n)}$, for $i=1, 2$. If $\phi = a_1\phi_1 + a_2\phi_2 (|a_1|^2+|a_2|^2=1,\quad a_i\neq 0)$ then for $\mathbf{F} = \mathbf{U}(\mathbf{P}_{[\phi]}\otimes\mathbf{W}_a)\mathbf{U}^{-1}$ we have

$$\mathbf{F} = \sum w_n \mathbf{P}_{[\beta_n]}\,, \tag{10}$$

where $\beta_n = \mathbf{U}(\phi\otimes\gamma_n) \in \mathbf{H}_o\otimes\mathbf{H}_{\mu(n)}$.

But

$$\mathbf{U}(\phi\otimes\gamma_n) = a_1\mathbf{U}(\phi_1\otimes\gamma_n) + a_2\mathbf{U}(\phi_2\otimes\gamma_n)\,, \tag{11}$$

and (10) cannot hold unless $\mu^1(n) = \mu^2(n) = \mu(n)$ for all n. If these identities were to obtain, however, then although $\mathbf{P}_{[\phi_1]}$, $\mathbf{P}_{[\phi_2]}$ are $\mathbf{O}$-distinguishable, $\mathbf{F}_1$ and $\mathbf{F}_2$ would be $(\mathbf{I}\otimes\mathbf{A})$-indistinguishable. For

$$\mathrm{tr}(\mathbf{F}_i\mathbf{I}\otimes\mathbf{A}_m) = \sum w_n \tag{12}$$

(over all n such that $\beta_{in} \in \mathbf{H}_o\otimes\mathbf{H}_{\mu m}$).

It follows from (9) that if $\mu^1(n) = \mu^2(n)$ for all n, then

$$\mathrm{tr}(\mathbf{F}_1\mathbf{I}\otimes\mathbf{A}_m) = \mathrm{tr}(\mathbf{F}_2\mathbf{I}\otimes\mathbf{A}_m) \tag{13}$$

for all m, and hence that $\mathbf{U}$ is not a $\mathbf{W}_a$ measurement.

V. APPROXIMATE SOLUTIONS

Some investigators have responded to the problem of measurement by devising an approximate treatment of measurement that seeks to take into account the macroscopic nature of the apparatus.[9–11]

The basic idea behind these proposals is to show that the final state $\mathbf{F}$ of the joint object-apparatus system that emerges from measurement is approximately given by a density operator $\mathbf{F}'$ that is a mixture of states in each of which the apparatus observable has a definite value. Thus although the negative solution to the problem of measurement shows that strictly speaking $\mathbf{F}\neq\mathbf{F}'$, nevertheless one wants $\mathbf{F}\simeq\mathbf{F}'$. Such proposals raise questions concerning the nature of the suggested approximation and concerning the class of measurements that will exhibit these approximations. Weidlich[12] suggests that the approximations satisfy the following requirements:

$$\mathbf{F} = \mathbf{F}' + \mathbf{X} \tag{14}$$

and for all states ψ,

$$(\mathbf{X}\psi,\psi) \ll (\mathbf{F}'\psi,\psi)\,. \tag{15}$$

Since

$$0 \leq (\mathbf{F}'\psi,\psi) \leq 1\,, \tag{16}$$

condition (15) might be construed as requiring that $(\mathbf{X}\psi,\psi)$ be negligible compared with $(\mathbf{F}'\psi,\psi)$, provided it were the case that for all ψ

$$(\mathbf{X}\psi,\psi) \geq 0\,. \tag{17}$$

But since $\mathbf{X} = \mathbf{F} - \mathbf{F}'$ and $\mathrm{tr}(\mathbf{F}) = \mathrm{tr}(\mathbf{F}') = 1$, one has

$$\mathrm{tr}(\mathbf{X}) = 0\,. \tag{18}$$

But (17) and (18) imply that

$$\mathbf{X} = 0\,, \tag{19}$$

i.e., that $\mathbf{F} = \mathbf{F}'$ which, by the results of Sec. VI, we know cannot be the case. It follows that (17) will not hold for every state vector ψ. Thus condition (15) that Weidlich imposes on the approximation cannot guarantee that the difference between $\mathbf{F}$ and $\mathbf{F}'$ is negligibly small.

[8] The preceding account is based on the assumption that **0** has a discrete spectrum. It does, however, allow for multiplicity. See Ref. 7.

[9] A. Daneri, A. Loinger, and G. M. Prosperi, Nucl. Phys. **33**, 297 (1962); Nuovo Cimento **44B**, 119 (1966).

[10] J. Bub, Nuovo Cimento **57B**, 503 (1968), is a careful critique of the proposals of Ref. 9.

[11] W. Weidlich, Z. Physik **205**, 199 (1967); also F. Haake and W. Weidlich, *ibid.* **213**, 451 (1968).

[12] W. Weidlich, Ref. 11, p. 208.

Since quite generally there will be ψ in $\mathbf{H}$ for which the difference between $\mathrm{tr}(\mathbf{P}_{[\psi]}\mathbf{F})$ and $\mathrm{tr}(\mathbf{P}_{[\psi]}\mathbf{F}')$ is not negligible, if one still wants to maintain that in some sense $\mathbf{F}\simeq\mathbf{F}'$, then clearly one must hold that such a ψ (or the corresponding observable $\mathbf{P}_{[\psi]}$) is "inadmissible." Quite generally, therefore, proposals for an approximate solution to the problem of measurement would seem to involve the following elements. Select a set $\mathcal{S}$ of observables on $\mathbf{H}$ and define

$$\mathbf{F}\simeq\mathbf{F}' \ (\mathrm{mod}\ \mathcal{S}) \text{ if and only if } \mathrm{tr}(\mathbf{SF})=\mathrm{tr}(\mathbf{SF}') \quad \text{for all } S\in\mathcal{S}.^{13} \tag{20}$$

The sense of the approximation would then be that one cannot discriminate between $\mathbf{F}$ and $\mathbf{F}'$ by computing the average value of observables in $\mathcal{S}$. Presumably one would want this to guarantee that no observations whatsoever could discriminate between $\mathbf{F}$ and $\mathbf{F}'$.[14,15] It should be apparent that this guarantee cannot be provided by observations, experimental evidence, or the like. For the issue here is precisely to determine how to make the connection between theory and observation. Before we have settled on how that connection is to be made, we are not in a position to bring observational evidence to bear on matters theoretical. I suggest, therefore, that proposals for an approximate solution to the problem of measurement are in fact proposals for changing the usual interpretative rules of quantum theory.[16] More precisely, they propose that states $\mathbf{F}$ and $\mathbf{F}'$, which are held to be distinct on the usual understanding of quantum theory, should be considered as having the same observational significance; namely, they are both theoretical expressions of the situation which obtains when the joint system is such that the apparatus observable $\mathbf{A}$ does have one of its values and indeed has the value μ_n with probability $\mathrm{tr}(\mathbf{I}\otimes\mathbf{A}_n\mathbf{F})=\mathrm{tr}(\mathbf{I}\otimes\mathbf{A}_n\mathbf{F}')$. To understand proposals for approximate solutions in this way is not, of course, to forswear the supporting of such proposals. It does, of course, locate the type of support that is appropriate; viz., support in terms of consistency, economy, scope, simplicity, and the like.

Let me conclude this section with some remarks on how approximate solutions may go.

(a) Let $\mathcal{S}$ be the class of observables $\mathbf{S}=\mathbf{I}\otimes\mathbf{B}$ for $\mathbf{B}$ on $\mathbf{H}_a$ such that for some measurable function $\mathbf{f}$, $\mathbf{f}(\mathbf{A})=\mathbf{B}$. Then $\mathbf{F}\simeq\mathbf{F}'$ (mod $\mathcal{S}$). For then $\mathbf{B}=\sum\mathbf{f}(\mu_n)\mathbf{A}_n$ is the spectral resolution of $\mathbf{B}$, and

$$\begin{aligned}\mathrm{tr}(\mathbf{SF})&=\sum\mathbf{f}(\mu_n)\ \mathrm{tr}(\mathbf{I}\otimes\mathbf{A}_n\mathbf{F})\\&=\sum\mathbf{f}(\mu_n)\ \mathrm{tr}(\mathbf{I}\otimes\mathbf{A}_n\mathbf{F}')=\mathrm{tr}(\mathbf{SF}').\end{aligned} \tag{21}$$

The scope of this approximation is, however, very limited since not all the physically significant observables on $\mathbf{H}_a$ are functions of $\mathbf{A}$.

(b) It is sometimes suggested that for a macroscopic system there can be no incompatible observables; i.e., that all the observables commute. In line with this suggestion, one might like to enlarge the above example by letting $\mathcal{S}$ be the class of observables $\mathbf{S}=\mathbf{I}\otimes\mathbf{B}$ where $\mathbf{BA}=\mathbf{AB}$. Unfortunately, one no longer has $\mathbf{F}\simeq\mathbf{F}'$ (mod $\mathcal{S}$). Sufficient for this approximation to obtain, however, is that $\mathbf{A}$ has a multiplicity-free spectrum. On this assumption, if $\mathbf{B}=\sum\alpha_n\mathbf{B}_n$ is the spectral resolution of $\mathbf{B}$ then we can write each $\mathbf{B}_n$ as a sum of the 1-dimensional projections $\mathbf{A}_n$. Since one has $\mathrm{tr}(\mathbf{I}\otimes\mathbf{A}_n\mathbf{F})=\mathrm{tr}(\mathbf{I}\otimes\mathbf{A}_n\mathbf{F}')$ for every n, it follows that $\mathrm{tr}(\mathbf{I}\otimes\mathbf{B}_n\mathbf{F})=\mathrm{tr}(\mathbf{I}\otimes\mathbf{B}_n\mathbf{F}')$ for every n and hence that $\mathrm{tr}(\mathbf{I}\otimes\mathbf{BF})=\mathrm{tr}(\mathbf{I}\otimes\mathbf{BF}')$.

(c) Both (a) and (b) restrict the admissible observables to those of the form $(\mathbf{I}\otimes\mathbf{B})$. Let $\mathcal{S}$ be the class of all such observables; then it would be of interest to characterize the class of $\mathbf{W}_a$ filters $\mathbf{U}$ such that $\mathbf{F}=\mathbf{U}(\mathbf{W}_o\otimes\mathbf{W}_a)\mathbf{U}^{-1}$ and the corresponding $\mathbf{F}'$ satisfy $\mathbf{F}\simeq\mathbf{F}'$ (mod $\mathcal{S}$). In this regard one might notice that any von Neumann-type measurement yields this approximation.[17]

(d) There is, finally, some question concerning the time stability of these approximations. If $\mathbf{F}\simeq\mathbf{F}'$ (mod $\mathcal{S}$) at time t_0 (i.e., for $\mathbf{U}_{t_0}$), then if the joint system is left to evolve without interference (i.e., according to the dynamical group $t\rightarrow\mathbf{U}_t$) will the approximation continue to hold for times $t>t_0$? Thus if (20) holds, need $\mathbf{U}_t\mathbf{F}\mathbf{U}_t^{-1}\simeq\mathbf{U}_t\mathbf{F}'\mathbf{U}_t^{-1}$ (mod $\mathcal{S}$)?

This requires

$$\mathrm{tr}(\mathbf{S}\mathbf{U}_t\mathbf{F}\mathbf{U}_t^{-1})=\mathrm{tr}(\mathbf{S}\mathbf{U}_t\mathbf{F}'U_t^{-1}) \tag{22}$$

for all $\mathbf{S}\in\mathcal{S}$. Commuting we get, equivalent to (22),

$$\mathrm{tr}(\mathbf{U}_t^{-1}\mathbf{S}\mathbf{U}_t\mathbf{F})=\mathrm{tr}(\mathbf{U}_t^{-1}\mathbf{S}\mathbf{U}_t\mathbf{F}'). \tag{23}$$

If $\mathcal{S}$ is closed under the dynamical group, that is, if

$$S\in\mathcal{S} \text{ implies that } \mathbf{U}_t^{-1}\mathbf{S}\mathbf{U}_t\in\mathcal{S}, \tag{24}$$

then (23) follows from the approximation condition $\mathbf{F}\simeq\mathbf{F}'$ (mod $\mathcal{S}$).

It seems reasonable to expect that if $\mathcal{S}$ is the family of all "physically significant" observables on $\mathbf{H}$, then $\mathcal{S}$ is closed under the dynamical group and hence that approximations modulo $\mathcal{S}$ are stable in time. It should be noted, however, that the families $\mathcal{S}$ of (a)–(c) above are not closed.

VI. DISCUSSION

I have tried to show that there is a problem about quantum measurements that cannot be resolved within the usual theoretical framework. The upshot of this

[13] J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1968), Chap. 11, calls this relation "equivalence with respect to $\mathcal{S}$."

[14] This is the gist of Jauch's way of dissolving the problem of measurement. See Ref. 13, and also J. M. Jauch, E. P. Wigner, and M. M. Yanase, Nuovo Cimento **48**, 144 (1967).

[15] K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966), Vol. I, Sec. 20, pp. 173–189.

[16] A. Peres and N. Rosen, Phys. Rev. **135**, B1486 (1964), make explicit the interpretative character of approximative solutions. Oddly, they would nevertheless seek to show the *experimental* indistinguishability of **F** and **F**′.

[17] For the notion of a von Neumann-type measurement, see Ref. 7, Sec. 4, pp. 119–121.

demonstration, if cogent, is very far reaching. For the crux of the demonstration is to show that according to theory certain macroscopic observables, corresponding to the end product of a measurement (e.g., pointer positions), can take on no value whatsoever. The implication of this is that no laboratory observations can be cited in support of the quantum theory; e.g., the fact that an interference pattern emerges from a typical diffraction grating experiment is in *contradiction* with the theory. Surely, no one can take this seriously. The occurrence of an interference pattern, for example, is universally taken as supporting the theory. Thus in practice one treats the final superposed state of the joint object-apparatus system as having the same observational significance as some corresponding mixed state. The enormous difference between the two, which is the difference between the pointer aiming at some position or other in the mixed state but at no position at all in the superposed state, is treated as though it were no difference. This strategy of ignoring the difference is what I have referred to as an "approximate solution" to the measurement problem. It is the strategy adopted by all practitioners of the quantum theory, for it is the one that makes experimental support for the theory possible. There are, nevertheless, problems concerning the implementation of this strategy. These problems center around a precise formulation of alternative versions of the strategy, the scope and consistency of the interpretive rules embodied in these versions, and the type of support available for choosing one alternative over another. In Sec. V I have tried to lay the groundwork for discussing some of these issues. A clear and general statement of the proposed interpretive rules remains to be given.

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Examination of the "Leakage-Lifetime" Approximation in Cosmic-Ray Diffusion

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The diffusion of cosmic-ray particles in a finite volume of space with a simultaneous diffusion and/or transport in energy is considered. The solution of the appropriate differential equation may be expressed as an expansion in eigenfunctions of the differential operator. If one approximates the solution by keeping only the lowest or fundamental eigenfunction, one obtains the common "leakage-lifetime" approximation. In some situations this approximation can be justified, but in others (e.g., synchrotron or inverse-Compton losses) it cannot. The reason for the failure in this case can be seen from the point of view of the expansion. The solution of the general case of Fermi acceleration, synchrotron losses, and energy fluctuation acting together is also obtained by this method.

I. INTRODUCTION

THERE has been recent discussion in the literature as to the correct method of treating the loss of particles from a region of space where spatial diffusion and energy transport and/or diffusion are occurring simultaneously. A common method of treating this situation when it has arisen in the field of cosmic physics has been to describe it by an inhomogeneous, partial differential equation

$$\frac{\partial\rho(E,t)}{\partial t}+\mathcal{L}_E\rho(E,t)+\frac{\rho(E,t)}{\tau}=q(E,t). \qquad (1)$$

In Eq. (1), $\mathcal{L}_E$ is a differential operator in energy that describes the various energy-changing processes at work within the region, τ is the average lifetime of a particle against a variety of loss mechanisms, including leakage from the boundary, and $q(E,t)$ describes the energy distribution of the particles when they are introduced into the region; the inhomogeneous term q is often referred to as the injection spectrum.

Solutions of this equation are usually sought for the steady-state case $\partial\rho/\partial t=0$ for a variety of energy-transport mechanisms and injection spectra. In his now classic papers, Fermi[1,2] in essence solved this equation for the case $\mathcal{L}_E\rho=\partial(aE\rho)/\partial E$. In his first paper,[1] he considered $\tau=\tau_c$, the lifetime against nuclear collisions of the cosmic-ray particles. In his second paper,[2] he had come to the opinion that diffusive leakage from the galaxy was the most significant loss mechanism and hence considered $\tau=\tau_e$ or the "leakage lifetime." At present it is not believed to be very likely that Fermi's mechanism offers the correct explanation of cosmic rays; however, it is generally believed that he presented a correct treatment of a plausible process that should, in fact, occur even though it might not produce cosmic rays.

The time-independent form of Eq. (1) has been used extensively to calculate equilibrium spectra for a wide variety of problems in cosmic physics. Energy loss as

[1] E. Fermi, Phys. Rev. **75**, 1169 (1949).
[2] E. Fermi, Astrophys. J. **119**, 1 (1954).