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Mach's Critique of Newtonian Mechanics

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Mach's reinterpretation of Newtonian mechanics is examined and found ambivalent. His criticism of absolute time and absolute space was sound whereas his attempt to replace Newton's whole theory by an allegedly empirical statement and a definition was logically untenable. The whole program of reducing mechanics to kinematics is found to be unfeasible for logical reasons. The impossibility of deriving masses and forces from observations alone is discussed in detail. The driving force behind Mach's ideas on the foundations of mechanics is shown to be his empiricist philosophy which, while allowing him to impugn inscrutables, also suggested him to minimize the role of ideas. It is recalled that the discussion of the foundations of mechanics proceeds nowadays in a way far removed from Mach's operationalistic approach, namely via the axiomatic reconstruction of classical theories on the basis of undefined but meaningful concepts.

INTRODUCTION

A DISTINGUISHED experimental physicist and psychophysiologist, Ernst Mach (1838–1916) was an influential worker in the foundations of physics. His main contributions to the latter were critical rather than constructive, yet important because they were concerned with certain fundamental ideas. Many of Mach's criticisms, particularly those of atomistics and Newtonian mechanics, have had a profound influence. While the former was entirely negative, Mach's work on the foundations of mechanics was ambivalent.

On the one hand, Mach made a correct criticism of the concepts of absolute space and absolute time. On the other, he tried to minimize the role of mechanical theories and, indeed, of all theory. These two aspects of his work were as many consequences of the empiricist theory of knowledge he adopted and popularized. In fact, for a radical empiricist, every inscrutable idea is damnable and, moreover, no idea is to be accepted unless it concerns experience. By the first token, untestable assumptions are discarded from factual science, and rightly so. By the second, every theory is distrusted, or even condemned, because theories proper overreach experience and do not refer to it; and this was wrong.

Faithful to his empiricist philosophy, Mach attempted to cleanse physics from untestable hypotheses by the simple technique of eliminating hypotheses altogether—in fact by keeping only "experimental propositions" and definitions. Thus, his criticism of Newtonian mechanics was more of a criticism of theoretical physics than a criticism of classical physics—so much so that he attempted to replace that theory by a single empirical statement and a definition, and that he sanguinely opposed every attempt to go beyond classical physics, particularly relativity and atomic theories.[1] By criticizing Newtonian me-

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[1] See, e.g., E. Mach, *The Principles of Physical Optics* (1913; Dover Publications, New York, 1953), p. viii. For the opposite opinion, that Mach prepared the way to modern physics, see K. D. Heller, *Ernst Mach, Wegbereiter der Modernen Physik* (Springer-Verlag, Wien, 1964).

chanics, Mach undermined the prevailing dogmatism which regarded that theory as final, and thereby helped to create the freer climate of opinion necessary to invent new theories. But he rejected these new ideas because they were incompatible with his philosophy. Furthermore, it is a conceivable though untestable conjecture that the strictures imposed by Mach's narrow and dogmatic empiricism have delayed scientific progress, which does not consist in sticking to what has been experienced but in introducing daring new ideas enabling us to understand reality and to plan experiments unthinkable without those ideas.

Let us now examine the main features of Mach's criticism of Newtonian particle mechanics, the negative aspect of which has received more publicity than its positive aspect, to the extent that most "modern" textbooks on mechanics include Mach's blunders concerning mass and force.

I. THE RELATIVITY OF TIME

Mach rejects the Newtonian idea of absolute time which "of itself, and from its own nature, flows equably without relation to anything external."[2] He adopts instead what is properly called the relational and, improperly, the causal theory of time, which had been expounded by Plato, Aristotle, Lucretius, Leibniz, and others whom Mach does not name in this connection. (Mach himself acknowledged his ignorance of the history of philosophy. That he was a poor historian of mechanics is shown by the fact that he ignored the whole Middle Ages and—as shown by Truesdell[3]—that he drew from unreliable sources.)

According to the relational theory of time, time does not exist by itself but is something like the pace of events: there are changing things and "time is an abstraction at which we arrive by means of the changes of things."[4] The concept of absolute time, of a time independent of change, is "an idle metaphysical conception."[5] Like a good empiricist, Mach arrives at this conclusion by examining the way time is measured rather than the way the time variable occurs in theoretical physics. (This was also the approach of H. Reichenbach in his popular *Rise of Scientific Philosophy*.) From this examination he concluded that, "For the natural inquirer, determinations of time are merely abbreviated statements of the dependence of one event upon another, and nothing more."[6] In time measurements, one of the two processes being compared is "an arbitrarily chosen motion"[7] that functions as a time standard or clock.

Time, then, is relative; moreover it is local and even dispensable. In fact, although a time sequence can be set up for every part of the universe by means of some standard sequence of events, "For the universe [as a whole] there is no time."[8] In brief, there is local time but no cosmic time. (Consequently certain questions, such as the one concerning the age of the universe, are meaningless.) Moreover local time does not exist *per se* but is a relation among phenomena and can, so believes Mach, be eliminated. He thinks in fact that we could eliminate the time variable from any law statement by putting in its place a phenomenon dependent, say, on the angle of rotation of the Earth. (But he does not suggest how to implement this idea mathematically.) As can be seen, Mach's ideas on time were radical although they were neither too precise nor too new. Let us examine them briefly.

The idea that time is a relation between events rather than self-existent seems reasonable but is half-baked. Mach had in mind pairs of simultaneous events and was content with remarking that timing an event consists in pairing it to another event—its match in the standard sequence. This is true but insufficient: not only in science but also in modern everyday life we need a more complex concept of time, one with one foot on concrete events (whether actual or possible) and

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[2] I. Newton in *Mathematical Principles of Natural Philosophy*, F. Cajori, Ed. (University of California Press, Berkeley, 1947), p. 6.

[3] C. Truesdell, Arch. Hist. Exact Sci. 1, 3 (1960) and the Introduction to L. Euleri, *Opera omnia*, 11_2 (Soc. Sci. Natur. Helv., Zürich, 1960), especially p. 409 on Lagrange as Mach's historical misguide.

[4] E. Mach, *The Science of Mechanics* (Open Court Publishing Co., La Salle, Ill., 1942), 9th ed., p. 273.

[5] Reference 4, p. 274.

[6] E. Mach, *Popular Scientific Lectures* (Open Court Publishing Co., Chicago, 1910), 4th ed., p. 204.

[7] Reference 4, p. 275.

[8] E. Mach, *History and Root of the Principle of Conservation of Energy* (Open Court Publishing Co., Chicago, 1911), p. 63.

another on pure number. We need, in short, a quantitative concept of duration if we are to have physical theory in addition to the empirical recognition of equal durations.

A more sophisticated relational concept of time can be built in the following way. The lapse between two events x and y can be assigned a real number t. In general, letting E designate the set of point events and the Cartesian product $E\times E$ the set of ordered pairs $\langle x,y\rangle$ of events, the duration is a function D from $E\times E$ to T, the range T being included in the real line. In short, $D:E\times E\rightarrow T$ with $T\subseteq R$. An arbitrary member t of T can be called an instant of time. In order to determine the duration function D we must subject it to certain requirements (postulates). A possible set of conditions on D is this: (a) for every x in E, the duration of $\langle x,x\rangle$ is zero; (b) for any x and y in E, the duration of $\langle x,y\rangle$ equals, with changed sign, the duration of $\langle y,x\rangle$; (c) for any x,y, and z in E, the duration of $\langle x,y\rangle$ plus the duration of $\langle y,z\rangle$ equals the duration of $\langle x,z\rangle$. This theory of (nonrelativistic) time can be refined by introducing a third undefined physical concept in addition to those of event and duration, namely the one of reference frame. In any case it makes the relational or causal doctrine of time, adopted (though not invented) by Mach, somewhat more precise.

II. TIME STILL TIMELY

Yet, it is likely that Mach would have rejected the foregoing elucidation (not definition) of the time concept. First, because this analysis, far from eliminating concepts in favor of phenomena, shows that time is a certain relation (function) between pairs of events—whether phenomenal or imperceptible—and real numbers. Second, because it makes no reference to clock readings and moreover does not apply exactly to such empirical data. For these reasons the preceding analysis is not a formalization of Mach's ideas on time but rather a synthesis of the relational view of Leibniz and the Newtonian concept of cosmic time.

Moreover, the above theory of time does not enable us to eliminate the time concept, but rather reinforces it with the help of mathematics (more precisely, set theory). To see what would get lost if the time concept were eliminated, consider these two statements:

$$a \text{ happened on 1 January 1966,} \qquad (1)$$

$$b \text{ happened on 1 January 1966,} \qquad (2)$$

where a and b name certain events. From (1) and (2) and the definition of simultaneity we can eliminate the time datum, obtaining

$$a \text{ happened simultaneously with } b, \qquad (3)$$

$$b \text{ happened simultaneously with } a \qquad (4)$$

which, by virtue of the symmetry of the simultaneity relation, are equivalent. Yet, from the second pair of propositions we cannot reconstruct the first: the date has been lost and cannot be retrieved. And the date may be of importance; it will be indispensable if we expand the initial set of events. In conclusion, it would be foolish to eliminate the time concept: if we did it we would lose valuable information and we would be unable to build theories of change. Time must be analyzed, not eliminated. The same is true of the other abstractions condemned by Mach for being abstractions rather than for being wrong abstractions.

Let us finally comment on the alleged arbitrariness of the time standard. According to Mach "we select as our measure of time an arbitrarily chosen motion (the angle of the earth's rotation, or [the] path of a free body) which proceeds in almost parallel correspondence with our sensation of time."[9] This is false: (a) time standards are supposed to be regular rather than arbitrary and are corrected or replaced as soon as they are suspected of some irregularity; (b) since the beginning of scientific time reckoning the perception of duration has had little saying in the choice of standards; it has none in the adoption of "atomic" (actually molecular) clocks; (c) what is decisive in the adoption of a time standard is not our unreliable time sensation, which is rarely parallel to physical duration, but some theory which can justify the assumed regularity of the chosen process. Once again, the progress of science does not consist in sticking to sensations but in enriching our body of ideas and sub-

[9] Reference 4, pp. 275–276.

jecting sensations to their control as well as conversely.

In brief, Mach's criticism of the Newtonian concept of absolute time did contain a grain of truth—namely the old idea that time is relative. But this idea could not be developed along the lines of Mach's philosophy, which was diffident of mathematics: it was made precise with the help of mathematical concepts. And even this refinement proved eventually to be insufficient: Einstein (1905) showed that time is not only anchored in events but is also relative to the reference frame. But this further relativization of time was disowned by Mach.

III. THE RELATIVITY OF SPACE

Mach opposes Newton's idea of an absolute space existing "without relation to anything external."[2] He regards absolute space and absolute motion as "pure things of thought, pure mental constructs, that cannot be produced in experience."[10] Being transempirical they are to be condemned.

Mach was certainly right in rejecting the concepts of absolute space and absolute motion. But the philosophical reason he gave was wrong, as it boiled down to the requirement that physical theories be limited to human experience. He rightly remarked that the changes in position and velocity of a body b under the influence of another body b' can only be ascertained relative to a third material system. We may rephrase his criticism in the following way: free space has no intrinsic coordinate lines; if we wish to use the concept of coordinate system in physics we must at least imagine that it is the conceptual reconstruction of a material reference frame—a physical system, not a mathematical construct. Consequently, as far as physics is concerned all geometrical properties are determined with reference to physical systems, not to absolute space. These were not Mach's own words: his criticism, though correct, was not as clear as Leibniz's. Moreover, it was mixed with certain mistakes.

A first mistake was the failure to realize that C. Neumann (1870) had already supplied a substitute for Newton's absolute space—namely his Alpha body, a forerunner of our inertial or Galilean frame of reference. Mach regarded the Alpha body as a fiction[11] and instead advocated referring location and motion to the universe as a whole: "When . . . we say that a body preserves unchanged its direction and velocity *in space*, our assertion is nothing more or less than an abbreviated reference to *the entire universe.*"[12] This was a strange thing to say: (a) the universe, being known only in part, is not a suitable reference frame and is never used as such; (b) a good empiricist does not commit himself to assuming the objective existence of the universe—a realistic hypothesis. In any case, we can now reformulate Newtonian mechanics without either Newton's absolute space or Neumann's Alpha body or Mach's universe: one uses simply the set of inertial frames.

A second mistake of Mach's in relation with his criticism of the ideas of absolute space and absolute motion was that he thought position and motion to be *determined by* the reference frame rather than being *referred to* it. He failed to realize that a material frame of reference interacting significantly with the object of study is not a suitable frame, if only because it is not inertial. But he could not escape this confusion without leaping beyond his empiricist philosophy: in fact for an empiricist there is no determination apart from the operation whereby some human subject determines (perceives, measures, makes) properties of objects in his field of experience. In other words, an empiricist is bound to mix the ontological concept of determination (as exemplified by causation) with the epistemological concept of determination (as exemplified by measurement and prediction).

This confusion is at the root of the so-called Mach principle, according to which the motion and consequently (for Mach) the mass of every single body is determined (*bestimmt* = caused = produced) by the remaining bodies in the universe.[13] This "principle," the child of a verbal confusion, has been restated in a number of ways, not all of them faithful to Mach. (E.g., the hypothesis that the geometry of physical space is determined by matter.) Yet, though it has not been stated unambiguously, it passes for having

[10] Reference 4, p. 280.

[11] Reference 4, p. 291.

[12] Reference 4, p. 286.

[13] Reference 4, p. 283.

been confirmed by experiment, and it is often invoked in gravitation theory, cosmology, and in elementary particle physics for no better reason than that it enjoys the alleged authority of Mach. Its spurious origin is not realized, nor is it realized that, being a causal hypothesis, it is inconsistent with Mach's proposal of replacing causation by functional interdependence.[14] Furthermore, if the Mach *principle* were true, the mass of the universe should be zero, for it interacts with nothing.

This is not all: Mach stretched his criticism of the ideas of absolute space and absolute motion to the point of claiming that the heliocentric (Copernican) and the geocentric (Ptolemaic) "systems of the world" are equivalent: "the motions of the universe are the same whether we adopt the Ptolemaic or the Copernican mode of view. Both views are, indeed, equally *correct*; only the latter is more simple and more *practical*."[15] This thought has been repeated *ad nauseam* ever since. It is a favorite with the positivists in their criticism of realism; with the conventionalists and pragmatists in their advocacy of simplicity as a substitute for truth; last but not least with those who wished to rehabilitate the Inquisition and blame Galilei. Yet, notwithstanding the authority of Mach, Duhem, Pearson, Le Roy, Frank, Reichenbach, Goodman, and hundreds of textbook authors, a relativistic view of space and motion does not warrant the grotesque claim that the heliocentric and the geocentric "systems of the world" are equivalent except geometrically, in the sense that any trajectory in one of the "systems" can be transformed to the other.

Among the many reasons for the physical inequivalence between the two doctrines, the following can be mentioned. (a) Planets are not inertial systems, whereas the sun is nearly inertial; equivalently: the laws of mechanics do not hold exactly if referred to the earth but they hold approximately if referred to the sun—hence the heliocentric "system" is the one recommended by mechanics. (b) The solar gravitational field, which codetermines the planetary motions, cannot be transformed away—except locally—either in classical physics or in general relativity; equivalently: the equivalence principle is a local, not an integral law. (c) The Copernicus–Kepler model of the planetary system, unlike the Ptolemaic model, can be embedded in a number of physical theories, and in this way numerous phenomena can be explained—e.g., the stellar parallax, aberration, the Doppler shift, the irregularities in the planetary motions—which remain separate and mysterious on the geocentric doctrine. For some of these and other reasons Poincaré, who had initially embraced the view that the two "systems of the world" are equivalent, on re-examining it concluded that "the truth for which Galilei had suffered is still the truth."[16]

In short, Mach's critique of absolute space and absolute motion was basically correct but it was mingled with several important mistakes, among them the confusion of "relation to a frame" with "determination by a frame," and the purely kinematical conception of motion, with oblivion of the force fields that can be involved in it. This latter mistake enabled Mach to regard force and mass as dispensable. To this point we now turn.

IV. THE ELIMINATION OF MASS AND FORCE

Mach disliked the concepts of mass and force, which were at the heart of the popular materialist philosophy he opposed. He objected, in particular, to the prevailing idea that mass was a measure of the quantity of matter—a concept he could not accept. Indeed, since matter is by definition supposed to exist by itself, it should not be characterized by reference to sense impressions alone, and is therefore unfit to enter an empiricist schema, in which existence must be identified with capacity to have or to produce sense impressions—Berkeley's famous "To be is to perceive or to be perceived." Science, according to Mach, must endeavor to record phenomena (observable facts) and "to establish the interdependence of phenomena." Since the phenomena of mechanics are motions of perceptible bodies, we must limit our attention to the kinematics of bodies. This had also been d'Alembert's program in his influential *Traité de dynamique* (1743).

[14] M. Bunge, *Causality: The Place of the Causal Principle in Modern Science* (The World Publishing Company, New York, 1963), 2nd ed.

[15] Reference 4, p. 284.

[16] H. Poincaré, *Science et hypothèse* (Librairie Ernest Flammarion, Paris, 1902), p. 274.

To implement the kinetic approach in the case of the mass concept, Mach considered the following example. Imagine an isolated system consisting of two bodies "perfectly equal in all respects." (One ought to qualify that these traits are observational, for in order to substantiate the hypothesis of identity one would have to measure, among other things, the masses of the bodies.) He expects, and finds by experiment, that "they will produce in each other in the direction of their line of junction equal and opposite accelerations." And this, according to Mach, is all we mean when we say that the bodies concerned have the same mass. In fact, he proposed "arbitrarily establishing the following definition: All those bodies are bodies of equal mass which, mutually acting on each other, produce in each other equal and opposite accelerations."[17]

Thinking that he had *defined* the concept of mass equality and, indeed, without the help of Newton's *lex secunda*, he went on to metricize the mass concept by considering a slightly different example: a system composed of two arbitrary bodies which may be called 1 and 2. A comparison of their mutually induced accelerations $-a_1$ and $+a_2$ enables him to measure or else to compute their mass ratio. (Let us add that a suitable reference frame must be chosen.) And once he has confused "measuring" and "computing" with "defining" he can conclude that he has *defined* the mass concept in terms of observable (kinematic) properties. (A similar confusion lies at the root of Bridgman's operationalism.) In fact, pretending to ignore Newton's second and third axioms, Mach feigns that "experience alone can teach" that the ratio $-a_2/a_1$ is constant in time and positive, i.e., $-a_2/a_1=m_{12}$, where m_{12} is just an abbreviation for that ratio. He finally writes the equation in the form

$$-a_2/a_1=m_1/m_2, \qquad (5)$$

which is just a way of stating the "fact" that the ratio of accelerations is constant. (Mach did not inquire whether it remains constant under transformations of reference frames, in particular whether $-a_2/a_1$ is the same relative to an accelerated frame. In his piece-meal approach that jumps from one example to the next he seems to have forgotten the relativity of motion.)

[17] Reference 4, p. 266.

Equation (5) was taken by Mach as the *definition* of relative mass, or mass ratio—namely as "the negative inverse ratio of the counter-accelerations."[17] Once the mass concept was so "defined" he turned to the concept of force, which he "defined" as $m\cdot a$.[18] Consequently, Newton's *lex secunda*, far from expressing a law of nature, is degraded to a convention.

With this simple device, Mach believed to have attained his goal—"to remove all metaphysical obscurity"[19]—since mass had now become an observable property of observable bodies and force a mere name for the product of two observable properties. (Atoms were considered by Mach as annoying fictions precisely because they were hypothesized to be beyond sense experience. Hence there was no need to bother about atomic masses.) Moreover, the principle of action and reaction being absorbed in Eq. (5), it need not be stated separately. Mach did not recall at this point that the principle is also needed in statics, where no accelerations occur. And he was not aware that his formula, far from having been established by experiment alone, was a consequence of the second and third laws of motion. In fact, he claimed that "in the concept of mass no theory of any kind whatever is contained, but simply a fact of experience."[20] Which, though untrue, was a definite advantage from an empiricist point of view: one more theory could be dispensed with.

Actually Mach's "experimental proposition" (5) is not a direct expression of phenomena: from observation alone we could not infer that the components of a system move under the sole influence of their mutual actions: only an empirical checking of the consequences of such a *hypothesis*, when embedded in some theory, is possible. That proposition is in fact a deductive consequence of Newton's postulates for the very special case of an isolated system consisting of two interacting bodies. Being an application of a theory it does not allow us to dispense with the theory. It can moreover be anticipated that the extension or domain of applicability of

[18] Reference 4, p. 304.

[19] Reference 4, p. 267. One wonders what the reaction of Mach to the mass operators in elementary particle theory would have been.

[20] Reference 4, p. 271.

Mach's "experimental proposition" (5) is smaller than Newton's theory—i.e., that it will fail in more complex cases. This is seen in a moment.

V. MASS STILL IN FORCE

Mach's formula (5) is much weaker than the Newton–Euler theory of mechanics, so much so that it holds only for certain systems in equilibrium. The treatment of slightly more complicated problems requires the use of the full theory. Which was to be expected: no theory is equivalent to its lowest-level consequences, let alone to one or two of them. In other words, a theory is much more (actually infinitely more) than a set of examples. But then Mach had no appreciation for theoretical physics: for him the historical development of mechanics had established "only *one* great fact," namely the "experimental proposition" (5).[21]

But formula (5) breaks down when referred to an accelerated frame: in fact, relative to such a frame, a different acceleration ratio may obtain although the mass values are, by hypothesis, invariant under arbitrary changes in reference frames. To realize this, consider, again, Mach's example and compute the accelerations of its two constituents, first in a frame K linked to their center of mass—which is what Mach had unwittingly done—and then in another frame K' moving with accelerated motion relative to K and parallel to the line joining the two particles.[22] Calling a the acceleration of K' with respect to K, the new accelerations of the particles are

$$a_1' = a_1 - a, \tag{6}$$

$$a_2' = a_2 + a, \tag{7}$$

whence

$$\frac{a_1'}{a_2'} = \frac{a_1 - a}{a_2 + a} = \frac{(1 - a/a_1)}{(1 + a/a_2)} \cdot \frac{a_1}{a_2}. \tag{8}$$

If a_1/a_2 is, in fact, constant in time and positive, then Mach calls it $-m_{21}$ and interprets it as the mass of particle 2 relative to the mass of particle 1. If, in addition, the coefficient of a_1/a_2 is constant in time and positive, he calls the left-hand side of (8) $-m_{21}'$, whence the previous equation becomes, in Mach's interpretation,

$$m_{21}' = \frac{1 - a/a_1}{1 + a/a_2} \cdot m_{21}. \tag{9}$$

In particular, if K' moves alongside the first particle ($a = a_1$), then $m_{21}' = 0$ although $m_{21} \neq 0$. And by redoing the calculation for m_{12}' we can see that $m_{12}' = 0$ even though $m_{12} \neq 0$ if the acceleration of K' equals that of the 2nd particle. Which shows that the primed quantities m_{12}' and m_{21}' cannot be interpreted as the relative masses since in classical mechanics, and also in Mach's thought, these are invariant with respect to the choice of frame.[23]

To that physical mistake stemming from elaborating a single elementary exercise rather than elucidating the whole theory, Mach added one logical and one methodological mistake. The logical mistake was to confuse an equality with an identity and, in particular, with a definition. In fact, Mach's pseudopostulate (5), like most physical laws, establishes an equality between two expressions which differ in meaning and cannot therefore be regarded as the two sides of a definition. Indeed, while "m_1/m_2" means "the inertia of body 1 relative to the inertia of body 2," the symbol "$-a_2/a_1$" stands for a purely kinematical quantity. The equality is numerical not logical: it does not authorize us to eliminate one of the sides in favor of the other.

Similarly, it is mistaken to regard "$f = ma$" as a definition of force in terms of m and a. It is not just a question of calling ma by the name f or conversely: the two concepts happen to be related in that way in classical mechanics but there is nothing either necessary or conventional in this. If the *lex secunda* were just a convention it would be impossible to subject it to empirical tests; in particular, it would be impossible to find fault with it. Yet, we know it holds only relatively to inertial (Galilean) frames and, even so, only in the nonrelativistic limit and as a quantum-mechanical space average. Moreover, if it were a logical identity, masses and forces could not be assigned (hypothesized) independently. In particular, it would be impossible to assign masses to unaccelerated bodies.

[21] Reference 4, p. 307.
[22] See C. G. Pendse, Phil. Mag. **24**, 1012 (1937).
[23] C. G. Pendse, Phil. Mag. **27**, 51 (1939).

In conclusion, Mach's proposed redefinitions of mass and force are a failure from every possible point of view: they are not independent of mechanical theory, they are not generally applicable, and they do not qualify as definitions (logical identities). Their persistent lingering in textbooks may have to be explained as a result of philosophical loyalty and of the neglect of foundations research among physicists.

VI. THE KINETIC APPROACH

The philosopher-scientist René Descartes (1596–1650) was the first among the moderns to attempt to reduce the whole of physics to figures and motions, these ideas being "clear and distinct" unlike those of potency, act, substance, and accident employed by the schoolmen. Of course, Descartes did not succeed in carrying out this reduction program: he introduced invisible atoms and kept the Aristotelian aether, which he endowed with complex vortex motions. Moreover, being a deductivist not an inductivist, he had no intention of starting with observation: as every other theoretical physicist he started from assumptions that would eventually explain observations. His program, interesting as it was, was unsuccessful because he chose the wrong axioms and because he was more anxious to justify them by metaphysical argument than to put them to the experimental test. The universal genius Leibniz (1646–1716), similarly inspired, failed for the same reasons, although he was able to utilize a much more powerful mathematics.

Newton criticized Cartesian physics—without mentioning it by name—for being aprioristic. He proposed instead to build a theory that would start from observation and keep close to it: "In experimental philosophy we are to look upon propositions inferred by general induction from phenomena as accurately or very nearly true"; "the argument of induction may not be evaded by hypotheses."[24] Such the words not the deeds: Newton never practiced the inductivist philosophy he preached. In fact (a) he assumed objects that are not directly observable, such as mass, center of mass, and force; (b) he related such transempirical concepts to form hypotheses that he supposed stood for the invisible yet real patterns of motion, and (c) he assigned (hypothesized) forces and masses and worked out the consequences, which he finally checked against empirical data. In short, Newton was as much a practitioner of the hypothetico-deductive method as Descartes and Leibniz were—only he thought he was a faithful follower of Francis Bacon, the national philosophical hero.

Newton's anti-Newtonian philosophy was influential: people with no philosophical background thought Newton practiced what he preached. But he did not fool Kirchhoff and Mach, two inductivists who felt that Newton had overstepped the limits of experience. Without knowing it they tried to implement Descartes's kinetic program and the similar program of d'Alembert. Gustav Robert Kirchhoff (1824–1887), who seems to have exercised a strong influence on Mach and other contemporaries, wrote in his treatise: "In my opinion mechanics can create out of motion alone the definitions of the concepts it is concerned with."[25] That is, observable events (phenomena) are to be the building blocks of mechanics and, in general, of physics. Every other idea is to be regarded as an auxiliary construct, a useful fiction, as the philosopher Hans Vaihinger would put it later on. Thus, masses and forces are not to be found in reality, simply because they are not in human experience: they are just auxiliary concepts (*Hilfsbegriffe*).[26] These ideas, and the program of rescuing physics from the "metaphysical" ideas of Newton, had been pioneered by d'Alembert and have been in the air since the 1870's. Being the fashionable philosophy of physics, it has been adopted even by some of the creators of quantum mechanics—but only nominally, for when one speaks of a quantum-mechanical "observable" one means a symbolic and high-brow representative of some physical property not directly accessible to the senses.

Physicists cannot work in the way recommended by inductivists: they cannot help introducing transobservational concepts and they do not deduce theories from observations because this would be logically miraculous. In fact (a) experience does not supply those transobserva-

[24] Reference 2, p. 400.

[25] G. Kirchhoff, *Vorlesungen über mathematische Physik*, I, *Mechanik* (B. G. Teubner, Leipzig, 1883), p. 11.

[26] Reference 25, p. 1.

tional concepts; (b) particular propositions can be deduced from general ones but the converse is logically impossible. In particular, there is no unique path from observed motions to laws of motion: the latter are hypothesized, their solutions are then found, and finally these solutions are contrasted with empirical data obtained with the help of a bunch of theories. One and the same class of motions can be derived from infinitely many equations of motion. By contrast, given an equation of motion the class of motions remains uniquely determined. For example, in particle mechanics a uniform rectilinear motion can be derived from any of the following equations of motion:

$m\dot{x}=f=\text{const}$ (Aristotle's law for a constant force), (10)

$m\ddot{x}=0$ (Newton–Euler law for a null force), (11)

$m\ddot{x}=f+\lambda(\partial\varphi/\partial x)=0$ (Lagrange's law with a constraint balancing the applied force), (12)

$m\ddot{x}=f(t)+R(x,t)$, (13)

R being a random force whose space average balances the instantaneous applied force.

Not only the laws of motion and the forces have to be hypothesized, but also the masses have to be prescribed, if only tentatively, if the trajectories of definite systems are to be derived from the equations of motion. In some cases—e.g., elastic collision—the mass ratios can be inferred from data concerning particle velocities in conjunction with the law of conservation of linear momentum. In a few other cases the mass values can be inferred from observed accelerations and Newton's second and third laws. The latter method, advocated by Mach as a universal procedure, has severe limitations: as shown by Pendse,[22,23] the method works for a system consisting of at most 4 bodies when their accelerations are observed at one instant; and it works up to $n=7$ if observed at different instants.

Consider an isolated system containing $n>2$ particles. Call a_i the acceleration of the ith particle at a given instant (relative to a certain frame) and let a_{ik} be the contribution of the kth particle in the system to the acceleration of i. Calling e_{ik} the unit vector in the direction of i to k, we have

$$a_i=\sum_{k\neq i} e_{ik}\cdot a_{ik}. \tag{14}$$

Every such vector equation summarizes a system of 3 linear equations for the $(n-1)$ unknowns a_{ik} associated with the ith particle. For $n=4$ there will be as many equations as unknowns—save if the particles are collinear or coplanar, in which cases the number of equations will be less than that of unknowns. If $n>4$ the equations are less than the unknowns, which will therefore remain undetermined. In other words, the data a_i do not determine uniquely the relative accelerations a_{ik}, which are in turn proportional to the relative masses on condition that they are all referred to the same inertial frame. Similarly the method fails for $n>7$ observations taken at different instants. The only way to proceed is to single out a subsystem with $n<7$ particles or to assume that one of the components is dominant and apply a perturbation technique. But in such cases the computations are approximate, whence the mass ratios do not coincide with the acceleration ratios.

The unfeasibility of Mach's method becomes even more obvious in continuum mechanics, which, after all, is logically more basic and, of course, more realistic than particle mechanics. Indeed, consider, say, Cauchy's first law of motion,

$$\rho\ddot{x}=\text{div}T+\rho f_b, \tag{15}$$

where ρ, T, and f_b stand for the mass density, the stress tensor and the body force density, respectively, while x represents the position of an arbitrary particle in the body. Even assuming that the (infinitely many) accelerations were observable—which they obviously are not—nothing could be inferred from their knowledge alone. In practice, one hypothesizes ρ, f_b, and T, solves for x, and then compares the solutions with the observed motions. If ρ, f_b and T are given (hypothesized), the stress state is not uniquely determined by the three equations of motion, as T has nine components. If we want to solve any real problems we must hypothesize the occult T—e.g., by adding some constitutive equation involving T. And T is not an observable: it is a hidden property that manifests itself on the body boundary (as the stress vector $t = T \cdot n$). Not only in atomic physics but also in classical physics is the overt behavior of matter determined by hidden properties that are beyond observation although they do show up in a roundabout fashion. In the best of cases, the observable behavior can be calculated on the basis of additional assumptions concerning inner structures and states. The inverse way, from experiment to theory, simply does not exist. The way of science is not observation $\rightarrow$ theory but rather observation $\rightarrow$ problem $\rightarrow$ theory $\rightarrow$ experimental checking $\rightarrow$ theory correction.

In brief, in a few simple cases it is possible to infer mass values from observations fed to mechanical theory. But, in most cases, the mass values of the individual components of systems of bodies must be either hypothesized or just left as unknowns. Yet, although a mass value may not be computed from observed motions in most cases, it is assumed to be definite, i.e., to be an objective property of the body concerned. In other words, in mechanics we do not accept the positivist tenet that bodies acquire a mass upon being subjected to mass measurement operations but assume instead that they have a definite mass value all along. Very often we assign mass values hypothetically, deduce consequences, confront the latter with data and, if necessary, correct the hypothesized mass values by successive approximations. This would be impossible without assuming that "mass" refers to (stands for) an objective property—which is all Mach wished to deny.

VII. HOW NOT TO BUILD MECHANICS

The preceding considerations should suffice to show that the dynamics of a mechanical system can neither be ignored nor inferred from observations of the motions of every one of its parts—not even if these parts are observable. On the contrary, the motions are deducible from hypotheses concerning the laws of motion, the mass (or density) values, and the stresses and forces involved: the data only single out particular trajectories out of bundles of possible trajectories. In short, as Newton had discovered, kinematics is deducible from dynamics but not, as d'Alembert, Kirchhoff, and Mach wanted, conversely. In other words, the inverse problem of experimental mechanics—deriving masses, stresses and forces from a knowledge of motions alone—is in general as unsolvable as the problem of inferring postulates from theorems in a unique way.

Since this is true of the simplest possible physical systems—the mechanical ones—we may jump to the general conclusion that, while hypotheses concerning the nature of a system are, jointly with its laws of evolution, necessary and sufficient to determine its behavior in a broad way, given the behavior—in case it is observable—it is not possible to discover the nature and laws of the system. Consequently, every behavioristic (phenomenalist, empiricist) program, whether in mechanics or in elementary particles or in psychology, is bound to be superficial.[27] True, sometimes we are forced to remain on the surface: but let us not make a virtue of such a poverty.

Despite the logical impossibility of building mechanics—or for that matter any other scientific theory—with observational concepts alone; despite the logical impossibility of inferring theories from data; despite the sterility of inductivism; despite the successes of the more general and abstract formulations of mechanics; despite all this there have been further attempts to build mechanics on observational concepts. Those of Hermes[28], Simon,[29] and Eisenbud[30] are typical of that trend. Let us comment very briefly on the former, which is the earlier and the more elaborate.

The motivation of Hermes's work in the foundation of mechanics is the same as Mach's, namely the elimination of unobservables: since what we measure are values of kinematical magnitudes—e.g., the position of a pointer in the case of weight measurements with a scale —we should be able to get along with purely kinematical concepts. The following objections can be raised. (a) Every such measurement involves the use of a number of physical laws in

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[27] See M. Bunge, "Phenomenological Theories," in M. Bunge, Ed., *The Critical Approach to Science and Philosophy* (The Free Press of Glencoe, New York, 1964).

[28] H. Hermes, "Zur Axiomatisierung der Mechanik" (reformulation of a system published in 1938), in L. Henkin, P. Suppes and A. Tarski, Eds., *The Axiomatic Method* (North-Holland Publishing Co., Amsterdam, 1959).

[29] H. A. Simon, Phil. Mag. **38**, 888 (1947) and Phil. Sci. **21**, 340 (1954).

[30] L. Eisenbud, Am. J. Phys. **26**, 144 (1958).

which nonobservational concepts occur. (b) One of the characteristics of scientific theories is that they involve theoretical (nonobservational) concepts, this being why they can hope to explain experience. (c) The classical kinematics of the point particle is not a hypothetico-deductive system (theory) but a set of loosely related formulas: no wonder therefore that Hermes's system contains no law of motion and is therefore useless. (d) The concept of inertial or Galilean frame, which occurs as a primitive in this system, is not kinematical but dynamical, as it is defined in terms of Newton's laws of motion or some similar set. (e) The conservation of linear momentum, which Hermes postulates and from which he derives the mass concept, is a dynamical, not a kinematical law—and moreover it cannot function as a definition if it is to be a law of nature. (f) Even if in simple cases we can determine mass ratios from velocity ratios with the help of that law, such a numerical determination does not amount to a definition.

In conclusion, the kinetic approach, which would reduce everything to observable figures and motions, was made obsolete by Newton three centuries ago. Moreover, it cannot be implemented, for if nothing is assumed in addition to what is observed, then nothing can be deduced. And if there is no deduction, there is no theory—which may be all right for empiricist philosophers but is lethal for science.

The morals for theory construction are clear: (a) Do not try to define the basic concepts, as they will be needed to define all others; (b) do not start with observables, otherwise you will be unable to explain them; (c) do not start with weak assumptions (theorems) unless you are willing to pile up infinitely many of them. In particular, in reconstructing classical mechanics one starts with a bunch of primitive (undefined) concepts such as those of body, configuration, mass, and force, and lays down basic requirements (axioms) to be satisfied by the primitives. Such a reconstruction is under way, notably thanks to Suppes[31] and Noll.[32] Strange as it may seem, they are quite recent and are systematically ignored by those who think that Mach's foundational work was final or nearly so—although it was done at a time when mathematical logic was an esoteric science and scientific semantics had not been born.

The modern treatment of mechanics is not finished either. The available axiomatizations of mechanics are formally correct but they limp on the semantical side, in the sense that they correct the operationalist mania of attaching to every sign an operational meaning by failing to assign any objective physical meaning at all. The construction of formally correct and semantically adequate axiomatizations of the various branches of physics—not excluding mechanics—is an unfulfilled task of foundational research.[33] It does not need the help of automatic computers, and it will not lead to designing more effective lasers but it should contribute to clarify ideas that, though capital and ever inspiring, are still half-baked or obscure.

VIII. CONCLUDING REMARKS

We have examined Mach's views on Newtonian mechanics and some of its descendants because, although they contained grave mistakes, they have contributed to free many minds from a blind faith in the Newton–Euler theory at a time when it seemed final. An additional reason for our examination is the amazing fortune Mach's mistakes have had with textbook writers (who, on the other hand, usually ignore Mach's arguments against absolute time and absolute space). In fact, although Mach's ideas do not occur in the great treatises on mechanics—such as those of Levi–Civita, Whittaker, and Truesdell—hundreds of textbooks follow Mach in trying to reduce dynamics to kinematics by eliminating the concepts of mass and force via definitions in terms of observational concepts. Of course, they just try. They do not realize that Mach's was not just a reinterpretation of the Newton–Euler theory but its utter destruction through its replacement by two formulas, the allegedly experimental proposition (5)—claimed to be at the same time a law and a definition

[31] J. C. C. McKinsey, A. C. Sugar, and P. Suppes, J. Rat. Mech. Anal. **2**, 253 (1953).

[32] W. Noll, "The Foundations of Classical Mechanics in the Light of Recent Advances in Continuum Mechanics," in the collective work cited in Ref. 28.

[33] An attempt to fill formalisms with a physical content while respecting the axiomatic method is made in the author's forthcoming *Foundations of Physics* (Springer-Verlag, Berlin, to be published).

—and the "definition" of force. And they do not know that the question whether a statement is a definition or a hypothesis is no longer a matter of opinion but can be settled in an exact way by means of a logical technique (Padoa's method[34]) as long as it is applied in a closed context (an axiomatized theory).

Mach's mistakes were not accidental: they were due to his philosophy and to his inadequate knowledge of theoretical physics. (Nowhere in his work are four of the Big Five of theoretical physics in the 19th century mentioned: Cauchy's theory of elasticity, Maxwell's electromagnetism, Maxwell–Boltzmann's statistical mechanics, and Lorentz's electron theory. He only mentioned Fresnel's mechanical theory of light.) Mach was one of the very few scientists who evolved explicitly his philosophical ideas and tried to reform physics by adapting it to his philosophy, which continued the sensism of the 18th-century British empiricists Hume and Berkeley.[35] He thought, in particular, that all physical concepts should be clear (Cartesianism), and that complete clarity is attained through definition (Aristotelianism) in terms of sense data (sensism). In addition, Mach was unaware of the logical impossibility of defining every concept; he seemed to regard every equality as qualified for functioning as a definition; he did not realize that the basic ideas taken in isolation cannot help being somewhat obscure if they are rich and deep, precisely because they are basic (explanatory and definitory) and pregnant with a number of relations, many of which are unexpected; and he could not accept that, in order to be rich, an idea must be far from concrete experience: indeed he was not after conceptual richness at all but, like Kirchhoff before him, after a complete and simple description of experience (not of reality). In the light of this philosophy Mach was able to criticize certain inscrutable ideas, but at the same time he fostered an antitheoretical attitude which continues to this day under the name of operationalism.

Mach's mistakes in his criticism of Newtonian mechanics—his most distinguished contribution to foundations research—can be corrected with the assistance of a bit of logic, another of semantics, and a dose of realism. A critical study of Mach's work in the foundations of mechanics should be helpful, if only to avoid repeating his mistakes, which were those of a philosophy distrustful of ideas. Ignore all philosophy, and you will be the slave of one bad philosophy.[36]

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[34] See P. Suppes, *Introduction to Logic* (D. Van Nostrand, Princeton, N. J., 1957).

[35] For Berkeley as a precursor of Mach, and for a criticism of both, see K. R. Popper, Brit. J. Phil. Sci. **4**, 21 (1953); and *Conjectures and Refutations: The Growth of Scientific Knowledge* (Basic Books, Inc., New York, 1963), Chap. 3.

[36] For a comprehensive exposé of the philosophy of science, see the author's forthcoming *Scientific Research* (Springer-Verlag, Berlin, to be published, 1966), 2 vols.