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A Uniqueness Theorem for 'No Collapse' Interpretations of Quantum Mechanics

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We prove a uniqueness theorem showing that, subject to certain natural constraints, all 'no collapse' interpretations of quantum mechanics can be uniquely characterized and reduced to the choice of a particular preferred observable as determinate (definite, sharp). We show how certain versions of the modal interpretation, Bohm's 'causal' interpretation, Bohr's complementarity interpretation, and the orthodox (Dirac–von Neumann) interpretation without the projection postulate can be recovered from the theorem. Bohr's complementarity and Einstein's realism appear as two quite different proposals for selecting the preferred determinate observable—either settled pragmatically by what we choose to observe, or fixed once and for all, as the Einsteinian realist would require, in which case the preferred observable is a 'beable' in Bell's sense, as in Bohm's interpretation (where the preferred observable is position in configuration space).

1. The Interpretation Problem

On the orthodox (Dirac–von Neumann) interpretation[1] of quantum mechanics, an observable has a determinate (definite, sharp) value for a system in a given quantum state if and only if the state is an eigenstate of the observable. So, the orthodox interpretation selects a particular set of observables that have determinate values in a given quantum state; equivalently, a particular set of idempotent observables or propositions, represented by projection operators, that have determinate truth values. If the quantum state is represented by a ray or 1-dimensional projection operator e spanned by the unit vector $|e\rangle$, these are the propositions p such that $e \leq p$ or $e \leq p^{\perp}$ (where the relation '$\leq$' denotes subspace inclusion, or the corresponding relation for projection operators, and $p^{\perp}$ denotes the subspace orthogonal to p).

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[1] P.A.M. Dirac, *Quantum Mechanics* (Oxford: Clarendon Press, 1958), 4th edn, pp. 46–47; J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton: Princeton University Press, 1955), p. 253.

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The orthodox interpretation involves a well-known measurement problem (see Section 3), which Dirac and von Neumann resolve formally by invoking a projection postulate[2] that characterizes the 'collapse' or projection of the quantum state of a system onto an eigenstate of the measured observable. Dynamical 'collapse' interpretations of quantum mechanics[3] modify the unitary, Schrödinger dynamics of the theory to achieve the required state evolution for both measurement and non-measurement interactions, while retaining the orthodox criterion for determinateness.

'No collapse' interpretations avoid the measurement problem by selecting other sets of observables as determinate for a system in a given quantum state. For example, certain versions of the 'modal' interpretation[4] exploit the polar decomposition theorem to select a preferred set of determinate observables for a system S as a subsystem of a composite system $S+S^*$ in a state $e \in H \otimes H^*$. Bohm's 'causal' interpretation[5] selects position in configuration space as a preferred always determinate observable for any quantum state, and certain other observables are selected as inheriting determinate status at a given time from this preferred determinate observable and the state at that time. Bohr's complementarity interpretation[6] selects as determinate an observable associated with an individual quantum phenomenon manifested in a measurement interaction involving a specific classically describable experimental arrangement, and certain other observables inherit determinate status from this observable and the quantum state. We discuss these interpretations and the orthodox interpretation in Section 3.

There are restrictions on what sets of observables can be taken as simultaneously determinate without contradiction, if the attribution of determinate values to observables is required to satisfy certain constraints. The 'no-go' theorems for 'hidden variables' underlying the quantum statistics provide a

[2]See Dirac, *op. cit.*, p. 36; and von Neumann, *op. cit.*, pp. 351 and 418.

[3]E.g. D. Bohm and J. Bub, 'A Proposed Solution of the Measurement Problem in Quantum Mechanics by a Hidden Variable Theory', *Reviews of Modern Physics* **38** (1966), 470–469; G. C. Ghirardi, A. Rimini and T. Weber, 'Unified Dynamics for Microscopic and Macroscopic Systems', *Physical Review* **D34** (1966), 470–491.

[4]The idea of a modal interpretation of quantum mechanics was first introduced by van Fraassen. [B. van Fraassen, 'Hidden Variables and the Modal Interpretation of Quantum Statistics', *Synthese* **42** (1979), 155–165; 'A Modal Interpretation of Quantum Mechanics', in E. Beltrametti and B. van Fraassen (eds), *Current Issues in Quantum Logic* (Singapore: World Scientific, 1981), pp. 229–258. See also B. van Fraassen, *Quantum Mechanics: An Empiricist View* (Oxford: Oxford University Press, 1991)]. We have in mind here specifically the interpretations developed by Kochen and by Dieks. [S. Kochen, 'A New Interpretation of Quantum Mechanics', in P. Lahti, and P. Mittelstaedt (eds), *Symposium on the Foundations of Modern Physics* (Singapore: World Scientific, 1985), pp. 151–169; D. Dieks, 'Quantum Mechanics Without the Projection Postulate and Its Realistic Interpretation', *Foundations of Physics* **19** (1989), 1397–1423; D. Dieks, 'Modal Interpretation of Quantum Mechanics, Measurements, and Macroscopic Behavior', *Physical Review* **A49** (1994), 2290–2300.]

[5]D. Bohm, 'A Suggested Interpretation of Quantum Theory in Terms of 'Hidden Variables': Parts I and II', *Physical Review* **85** (1952), 166–179, 180–193. Different formulations of Bohm's theory treat observables such as spin differently. We discuss these differences in Section 3.

[6]N. Bohr, *Atomic Physics and Human Knowledge* (New York: Wiley, 1958).

series of such results. So, in fact, the options for a 'no collapse' interpretation of quantum mechanics in the sense we have in mind are rather limited. We begin with a brief review of some of these limiting results.

The constraint imposed by the Kochen and Specker theorem[7] requires that the values assigned to a set of mutually compatible observables, represented by pairwise commuting self-adjoint operators, should preserve the functional relations satisfied by these observables. For example, the constraint requires that the value assigned to an observable A should be the square of the value assigned to an observable B, if $A = B^2$. With sums and products defined for mutually compatible observables only, the observables of a quantum mechanical system form a partial algebra, and the idempotent observables form a partial Boolean algebra. Kochen and Specker show that a necessary condition for the assignment of values to all the observables of a quantum mechanical system, in such a way as to satisfy the functional relationship constraint, is the existence of an embedding of the partial algebra of observables into a commutative algebra; equivalently, the embedding of the partial Boolean algebra of idempotent observables into a Boolean algebra. A necessary and sufficient condition for the existence of an embedding of a partial Boolean algebra into a Boolean algebra is that, for every pair of distinct elements p, q in the partial Boolean algebra, there exists a homomorphism h onto the 2-element Boolean algebra $\{0, 1\}$ such that $h(p) \neq h(q)$.

As Kochen and Specker show, there are no 2-valued homomorphisms on the partial Boolean algebra of projection operators on a Hilbert space of three or more dimensions (much less any embeddings). For the homomorphism condition implies that for every orthogonal n-tuple of 1-dimensional projection operators or corresponding rays in H_n, one projection operator or ray is mapped onto 1 ('true') and the remaining $n-1$ projection operators or rays are mapped onto 0 ('false'). And this is shown to be impossible for the finite set of orthogonal triples of rays that can be constructed from 117 appropriately chosen rays in H_3: any assignment of 1s and 0s to this set of orthogonal triples satisfying the homomorphism condition involves a contradiction.

Now, an observable A can be compatible with an observable B and also with an observable C, while B and C are incompatible (represented by non-commuting operators). A and B will be representable as functions of an observable X, while A and C will be representable as functions of an observable Y, incompatible with X. If A denotes an observable of a system S, and B and C denote incompatible observables of a system S', space-like separated from S, then the functional relationship constraint, that the value of A as a function of X should be the same as the value of A as a function of Y, becomes a locality

[7] S. Kochen and E. P. Specker, 'The Problem of Hidden Variables in Quantum Mechanics', *Journal of Mathematics and Mechanics* **17** (1967), 59–87.

condition. Bell argued[8] that the general functional relationship constraint cannot be justified physically, but this weaker locality condition can. Bell's theorem[9] shows that the locality condition cannot be satisfied in general in a Hilbert space of four or more dimensions: there are sets of observables for which value assignments satisfying the locality condition are inconsistent with the quantum statistics. [More recent versions of Bell's theorem, e.g. Greenberger, Horne and Zeilinger (GHZ),[10] do not require statistical arguments, or minimize the statistics needed, e.g. Hardy[11]].

Several authors have considered the problem of constructing the smallest set of observables that cannot be assigned values in such a way as to satisfy the functional relationship or locality constraints. Kochen and Conway have reduced the number of rays in H_3 required to generate a contradiction from value assignments satisfying the Kochen–Specker homomorphism condition from 117 to 31.[12] Peres has shown how to derive a contradiction for a more symmetrical set of 33 rays in H_3, and for 24 rays in H_4.[13] Kernaghan has reduced Peres' 24 rays to 20.[14] Clifton has an eight ray Kochen and Specker argument in H_3 (but the proof requires quantum statistics to derive a contradiction).[15] Mermin proves a version of the Kochen and Specker theorem for nine observables in H_4, and a version of Bell's theorem (the GHZ version) for ten observables in H_8.[16] There are related results by Penrose and others.[17]

The question of how small we can make the set of observables and still generate a Kochen–Specker contradiction is important in revealing structural

[8]J. S. Bell, 'On the Problem of Hidden Variables in Quantum Mechanics', *Reviews of Modern Physics* **38** (1966), 447–475. Reprinted in *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press, 1987).

[9]J. S. Bell, 'On the Einstein–Podolsky–Rosen Paradox', *Physics* **1** (1964), 195–200. Reprinted in *Speakable and Unspeakable in Quantum Mechanics, op. cit.* Note that this article, published two years *before* the 1966 review article, was actually written after the review article.

[10]D. M. Greenberger, M. A. Horne and A. Zeilinger, 'Going Beyond Bell's Theorem', in M. Kafatos (ed.), *Bell's Theorem, Quantum Theory, and Conceptions of the Universe* (Dordrecht: Kluwer, 1989), pp. 73–76.

[11]L. Hardy, 'Quantum Mechanics, Local Realistic Theories, and Lorentz-Invariant Theories', *Physical Review Letters* **68** (1992), 2981–2984.

[12]The proof is unpublished. For a reference, see A. Peres, *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer, 1993), p. 197.

[13]A. Peres, *op. cit.*, Chap. 7.

[14]M. Kernaghan, 'Bell–Kochen–Specker Theorem with 20 Vectors', *Journal of Physics A: Math. Gen.* **27** (1994), L829.

[15]R. Clifton, 'Getting Contextual and Nonlocal Elements of Reality the Easy Way', *American Journal of Physics* **61** (1993), 443–447.

[16]N. D. Mermin, 'Hidden Variables and the Two Theorems of John Bell', *Reviews of Modern Physics* **65** (1993), 803–815. Note that the Mermin proofs, which are conceptually very simple, are not nearly so economical as the Peres proofs in terms of the number of rays.

[17]See H. Brown, 'Bell's Other Theorem and its Connections with Nonlocality. Part I', in A. van der Merwe, F. Selleri and G. Tarozzi (eds), *Bell's Theorem and the Foundations of Modern Physics* (Singapore: World Scientific, 1992), pp. 104–116 for an interesting discussion of 'no-go' theorems, especially the Mermin proofs, and Peres, *op. cit.*, p. 212, for an account of Penrose's 40-ray proof for a spin-3/2 system. See also J. Zimba and R. Penrose, 'On Bell Non-Locality Without Probabilities: More Curious Geometry', *Studies in History and Philosophy of Modern Physics* **24** (1993), 697–720.

features of Hilbert space, but of no immediate significance for a 'no collapse' interpretation of quantum mechanics. The relevant question in a sense concerns the converse issue. To provide a 'no collapse' interpretation of quantum mechanics in the sense we have in mind, we want to know how *large* we can take the set of determinate observables *without* generating a Kochen–Specker contradiction, i.e. we are interested in the maximal sets of observables that can be taken as having determinate (but perhaps unknown) values for a given quantum state, subject to the functional relationship constraint, or the maximal sets of propositions that can be taken as having determinate truth values, where a truth-value assignment is defined by a 2-valued homomorphism. (As we show below, even the orthodox interpretation selects such a maximal set.)

More precisely, the projection operators (or corresponding Hilbert space subspaces) of a quantum mechanical system form a lattice L that can be taken as representing the lattice of yes–no experiments or propositions pertaining to the system. We know that we cannot assign truth values to all the propositions in L in such a way as to satisfy the functional relationship constraint, or even the weaker locality condition, for all observables generated as spectral measures over these propositions. That is, we cannot take all the propositions in L as determinately true or false if truth values are assigned subject to these constraints. So the probabilities defined by the quantum state cannot be interpreted epistemically and represented as measures over the different possible truth value assignments to all the propositions in L. But we also know that any single observable can be taken as determinate for any quantum state (since the propositions associated with an observable generate a Boolean algebra), so we may suppose that fixing a quantum state represented by a ray e in H and an arbitrary preferred observable R as determinate places restrictions on what propositions can be taken as determinate for e in addition to R-propositions.

The natural question for a 'no collapse' interpretation of quantum mechanics would then appear to be: What are the *maximal* sublattices $D(e,R)$ of L to which truth values can be assigned, where each assignment of truth values is defined by a 2-valued homomorphism on $D(e,R)$, and the probabilities defined by e for mutually compatible sets of propositions in $D(e,R)$ can be represented as measures over the different possible truth value assignments to $D(e,R)$? As a further constraint, it seems appropriate to require that $D(e,R)$ is invariant under lattice automorphisms that preserve e and R (so that $D(e,R)$ is genuinely selected by e and R, and the lattice structure of H), and that $D(e,R)$ is unaffected if the quantum system S is regarded as a subsystem of a composite system $S+S'$, where S is not 'entangled' with S'. We shall refer to these sublattices as the 'determinate' sublattices of L.

If the aim is to exclude determinate values for observables (perhaps fixed by underlying 'hidden variables'), i.e. to prove a 'no-go' theorem, then the constraints on values should be as weak as possible. For our problem, however, we are interested in characterizing the maximal sublattices of L that *allow* an interpretation of the associated observables as determinate for a given quantum state. The constraints we are thereby led to impose reflect, roughly, the strongest 'classicality' conditions we can get away with, consistent with such an interpretation. So we require that $D(e,R)$ is a sublattice of L (rather than a partial Boolean subalgebra, in which operations corresponding to the conjunction and disjunction of propositions are defined only for compatible propositions represented by commuting projection operators), and that the possible truth value assignments are defined by 2-valued lattice homomorphisms (rather than 2-valued partial Boolean algebra homomorphisms, i.e. maps onto $\{0,1\}$ that reduce to 2-valued lattice homomorphisms only on each Boolean subalgebra of $D(e,R)$).

Note that the problem of characterizing the maximal sets of observables or propositions that can be taken as (simultaneously) determinate, without generating a Kochen–Specker contradiction, is not well-defined *unless* we impose constraints on the sets—there are clearly many different infinite sets of propositions that can be assigned determinate truth values without contradiction in this sense. The problem is only interesting relative to the requirement that a maximal set of determinate propositions is an extension of some physically significant algebraic structure of determinate propositions. Since the dynamical variables in classical mechanics are all simultaneously determinate, and any single observable in quantum mechanics can be taken as determinate, our proposal is to consider what part of the non-Boolean lattice of quantum propositions can be added to the Boolean algebra of propositions defined by the spectral measure of a particular quantum mechanical observable R, for a given quantum state e, before this extended structure becomes too 'large' to support sufficiently many truth value assignments, defined by 2-valued homomorphisms, to generate the quantum statistics for the propositions in the extended structure as measures over these different truth value assignments.

In the following section, we prove that the determinate sublattices $D(e,R)$ are uniquely characterized as follows:[18] In an n-dimensional Hilbert space H_n, suppose R has $m \leq n$ distinct eigenspaces r_i and the rays $e_{r_i} = (e \vee r_i^{\perp}) \wedge r_i$, $i = 1,...,$ $k \leq m$, are the non-zero projections of the state e onto these eigenspaces. The determinate sublattice $D(e,R)$ of L is then the sublattice $L_{e_{r_1}e_{r_2}...e_{r_k}}$ generated by the k orthogonal rays e_{r_i} and all the rays in the subspace $(e_{r_1} \vee e_{r_2} \vee ... \vee e_{r_k})^{\perp}$ orthogonal to the k-dimensional subspace spanned by the e_{r_i}. $L_{e_{r_1}e_{r_2}...e_{r_k}}$ can also be characterized as $\{e_{r_i}, i = 1,...k\}'$, the commutant in L of e_{r_i}, $i = 1,...k$, or as $\{p\colon e_{r_i} \leq p \text{ or } e_{r_i} \leq p^{\perp},\ i = 1,...k\}$.

[18]The uniqueness theorem supersedes earlier related results by one of us. See J. Bub, 'Quantum Mechanics Without the Projection Postulate', *Foundations of Physics* **22** (1992), 737–754; J. Bub, 'On the Structure of Quantal Proposition Systems', *Foundations of Physics* **24** (1994), 1261–1280; J. Bub, 'How to Interpret Quantum Mechanics', *Erkenntnis* **41** (1994), 253–273.

Physically, $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$ contains all those projections with values strictly correlated to the values of R when the system is in the state e. We note that the full set of (not necessarily idempotent) observables associated with $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$ includes any observable whose eigenspaces are spanned by rays in $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$. The set of maximal observables includes any maximal observable with k eigenvectors in the directions e_{r_i}, $i = 1,\ldots, k$.

Bell,[19] and also Bohm and Bub,[20] objected to the Jauch and Piron 'no-go' theorem,[21] which required that a truth value assignment h to the lattice of quantum propositions (equivalently, the probabilities assigned by dispersion-free states) should satisfy the constraint (a consequence of axiom 4° in their numbering) that

$$h(p \wedge q) = 1 \text{ if } h(p) = h(q) = 1$$

for any propositions p, q (even incompatible propositions represented by non-commuting projection operators). To reproduce the quantum statistics, the constraint should be required to hold only for expectation values generated by distributions over the hidden variables corresponding to quantum states, but not necessarily for arbitrary hidden variable distributions (in particular, not for the truth value assignments themselves).

Now, there exist 2-valued lattice homomorphisms on our determinate sublattices $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$, so axiom 4° holds for these sublattices. The assumption that fails is Jauch and Piron's axiom 5, which requires that any proposition not equal to the null proposition is assigned the value 1 (i.e. 'true') by some dispersion-free state. This is certainly a reasonable requirement on the full lattice L: if a proposition is assigned the value 0 by every dispersion-free state, it will have zero probability for every quantum state (represented as a measure over dispersion-free states), hence will be orthogonal to every quantum state, and so can only be the null proposition. However, a similar argument does not apply to interpretations that select proper sublattices of L. In fact, there are propositions in $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$ not equal to the null proposition (propositions in $(e_{r_1} \vee e_{r_2} \vee \ldots \vee e_{r_k})^{\perp}$) that have zero probability in the state e and are assigned the value 0 by all 2-valued homomorphisms on $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$. The reason this is possible is that $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$ is always a *proper* sublattice of L and is constructed on the basis of what the particular state e of the system is. So the Jauch and Piron argument for axiom 5 does not apply to the sublattices $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$.

Similarly, the Kochen and Specker argument that the existence of hidden variables requires a Boolean embedding of the full partial Boolean algebra of idempotent observables of a quantum mechanical system breaks down. If the

[19] J. S. Bell, 'On the Problem of Hidden Variables in Quantum Mechanics', *op. cit.*

[20] D. Bohm and J. Bub, 'A Refutation of the Proof by Jauch and Piron that Hidden Variables Can Be Excluded in Quantum Mechanics', *Reviews of Modern Physics* **38** (1966), 470-475.

[21] J. M. Jauch and C. Piron, 'Can Hidden Variables be Excluded in Quantum Mechanics?', *Helvetica Physica Acta* **36** (1963), 827–837.

lattice operations in $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ are confined to compatible elements (corresponding to commuting projection operators), then $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ can be regarded as a partial Boolean algebra. There exist 2-valued partial Boolean homomorphisms on $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$, in fact sufficiently many to generate the probabilities defined by the quantum state for the propositions in $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$, but insufficiently many to provide an embedding of $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ into a Boolean algebra. Our proposal is that the determinate sublattices $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$, for all R, provide a class of perfectly viable 'no collapse' interpretations of quantum mechanics, in which the functional relationship constraint is satisfied without a Boolean embedding.

In the final section, we show that the determinate sublattice of a composite system $S+M$ (representing a system and a measuring instrument), for the 'entangled' state of $S+M$ arising dynamically from a unitary evolution representing a quantum mechanical measurement interaction, contains the propositions corresponding to the S-observable correlated with the pointer observable of M, if the pointer observable (or some observable correlated with the pointer observable) is taken as the preferred determinate observable R. Note that this determinate sublattice of $S+M$ is derived *without any reference to measurement as an unanalyzed operation*, i.e. this natural description of the measurement process falls out as just a special instance of the linear Schrödinger quantum dynamics, without requiring the projection or 'collapse' of the quantum state to validate the determinateness of measured values, as in the orthodox interpretation. This is perhaps evident from the statement of the uniqueness theorem: $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ is generated from the non-zero Lüders projections of the quantum state e onto the eigenspaces of R. Different choices for the preferred determinate observable R correspond to different 'no collapse' interpretations of quantum mechanics. We shall illustrate this with the orthodox interpretation (without the projection postulate), the modal interpretation (in the versions developed by Kochen and by Dieks), Bohm's 'causal' interpretation (one natural way to develop an Einsteinian realism within quantum mechanics), and Bohr's complementarity interpretation. But first we turn to the theorem.

2. The Uniqueness Theorem

We begin with some definitions: Consider a composite quantum mechanical system $S^{**} = S+S^{*}$ represented on a Hilbert space $H^{**} = H\otimes H^{*}$. Suppose the state of S^{**} is represented by a ray $e^{**}\in H^{**}$. Let R,R^{*} and R^{**} denote (preferred) observables of S, S^{*} and S^{**} represented by self-adjoint operators defined on the Hilbert spaces H, H^{*} and H^{**}, respectively.

Definition 1. We define a *compound* observable, denoted by $R\&R^{*}$, as any S^{**}-observable on H^{**} whose eigenspaces are the tensor products of the eigenspaces of R and R^{*}. So the number of distinct eigenvalues of $R\&R^{*}$ is equal to the product of the number of distinct eigenvalues of R and the number of distinct eigenvalues of R^*. (For example, if H and H^* are both 2-dimensional and R and R^* each have two distinct eigenvalues, ± 1, then $R\&R^*$ is an observable in $H^{**} = H\otimes H^*$ with four distinct eigenvalues corresponding to four 1-dimensional eigenspaces, while the tensor product $R\otimes R^*$ has only two distinct eigenvalues, ± 1, corresponding to two 2-dimensional eigenspaces.)

Definition 2. We define *an observable induced by R^{**} on the subsystem S* as an observable R on H that exists if and only if there is an observable R^* defined on H^* such that $R^{**} = R\&R^*$. (It follows that R is unique up to a choice of eigenvalues, i.e. different induced observables share the same set of eigenspaces.)

Definition 3. We define *the state induced by e^{**} on the subsystem S* as the state represented by the ray $e\in H$ that exists if and only if there is a ray $e^*\in H^*$ such that $e^{**} = e\otimes e^*$.

Definition 4. We define *the restriction of $D^{**}(e^{**},R^{**})$ to H, denoted by $D(e^{**},R^{**})$*, as the set of all projection operators p on H such that $p\otimes I^*\in D^{**}(e^{**},R^{**})$, where I^* is the identity operator on H^*.

Theorem 1 (Uniqueness Theorem)

Consider a (pure) quantum state represented by a ray e in an n-dimensional Hilbert space H and an observable R with $m\leq n$ distinct eigenspaces r_i. Let $e_{r_i} = (e\vee r_i^{\perp})\wedge r_i$, $i = l, \ldots, k \leq m$, denote the non-zero projections of the ray e onto the eigenspaces r_i. Then $D(e,R) = L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$ is the unique maximal sublattice of $L(H)$ satisfying the following four conditions:

(1) Truth and probability (TP): $D(e,R)$ is an ortholattice admitting sufficiently many 2-valued homomorphisms, h: $D(e,R)\rightarrow\{0, 1\}$, to recover the joint probabilities assigned by the state e to mutually compatible sets of elements $\{p_i\}_{i\in I}$, $p_i\in D(e,R)$, as measures on a Kolmogorov probability space (X,F,μ), where X is the set of 2-valued homomorphisms on $D(e,R)$, F is a field of subsets of X, and

$$\mu(\{h\colon\ h(p_i) = 1,\ \text{for all}\ i\in I\}) = \mathrm{tr}(e\prod_{i\in I} p_i).$$

(2) R-preferred (R-PREF): the eigenspaces r_i of R belong to $D(e,R)$.
(3) e,R-definability (DEF): for any $e\in H$ and observable R of S defined on H, $D(e,R)$ is invariant under lattice automorphisms that preserve e and R.

(4) Weak separability (WEAK SEP): if H is a factor space of a tensor product Hilbert space $H^{**} = H \otimes H^{*}$, and the state e^{**} and preferred observable R^{**} on H^{**} induce the state e and preferred observable R on H, then $D(e,R) = D(e^{**},R^{**})$.

Remarks: The motivation for the conditions TP and R-PREF is clear from the preceding discussion. The condition DEF requires that the determinate sublattice is selected by the state e and a preferred observable R. The condition WEAK SEP is introduced to avoid a dimensionality constraint in the theorem: without this condition, the eigenspaces of R are required to be more than 2-dimensional. The idea behind WEAK SEP is simply that we want two systems that are not 'entangled' (and are endowed with their own preferred observables) to be separable, in the sense that each system is independently characterized by its own determinate sublattice of properties, where the determinate sublattice of a component system is the restriction of the determinate sublattice of the composite system to the component system. Put differently, the determinate sublattice of a model quantum mechanical universe should be unaffected if we add a system to the universe, and there is no entanglement arising from any interaction between the universe and the added system. The qualification 'weak' here is added to contrast the condition with Einstein's stronger separability requirement (see Section 3), that the determinate properties of two spatially separated systems should be independent of each other, even if the quantum state of the composite system is an 'entangled' state (a linear superposition of product states) resulting from a past interaction between the systems (as in the Einstein–Podolsky–Rosen argument).

The strategy of the proof of the theorem proceeds by showing that, if $p \in D(e, R)$, then for any e_{r_i}, $i = 1,..k$, either $e_{r_i} \leq p$ or $e_{r_i} \leq p^{\perp}$. Since $L_{e_{r_1}e_{r_2}\ldots e_{r_k}} = \{p: e_{r_i} \leq p \text{ or } e_{r_i} \leq p^{\perp},\ i = 1,\ldots k\}$ will be shown to satisfy the conditions of the theorem, maximality requires that $D(e,R) = L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$.

Proof: For any system S represented on a Hilbert space H, and any state $e \in H$ and preferred observable R on H, we can always choose a Hilbert space H^{*}, a state $e^{*} \in H^{*}$, and observable R^{*} on H^{*}, so that the composite system, $S+S^{*}$, is in the state $e \otimes e^{*}$ and the composite preferred observable is $R\&R^{*}$. Furthermore, we can always choose an R^{*} with at least one eigenspace, r^{*}_j, of dimensionality greater than two. By WEAK SEP, if $p \in D(e,R)$, then $p \otimes I^{*} \in D^{**}(e \otimes e^{*}, R\&R^{*})$, since $D(e,R)$ must be the restriction of $D^{**}(e \otimes e^{*}, R\&R^{*})$ to H. Even if an eigenspace r_i of R is 1-dimensional or 2-dimensional, the dimension of the eigenspace $r_i \otimes r^{*}_j$ in $H \otimes H^{*}$ is greater than two.

We shall show that if $p \in D(e,R)$, then for any e_{r_i}, $i = 1,\ldots k$, either $e_{r_i} \leq p$ or $e_{r_i} \leq p^{\perp}$, if $\dim(r_i)>2$. But the proof applies also to the determinate sublattice $D^{**}(e \otimes e^{*}, R\&R^{*})$. In this sublattice, $\dim(r_i \otimes r^{*}_j)>2$ even if $\dim(r_i) \leq 2$, so

$e_{r_i}\otimes e^*_{r^*_j}\leq p\otimes I^*$ or $e_{r_i}\otimes e^*_{r^*_j}\leq p^\perp\otimes I^*$ (where $e_{r_i}\otimes e^*_{r^*_j}$ is the orthogonal projection of $e\otimes e^*$ onto the ijth eigenspace of $R\&R^*$). It follows that $e_{r_i}\leq p$ or $e_{r_i}\leq p^\perp$, even if $\dim(r_i)\leq 2$.

So suppose $p\in D(e,R)$. Consider the k eigenspaces r_i of R on which e has a non-zero projection. Since these eigenspaces are in $D(e,R)$ (by R-PREF) and are assigned non-zero probability by the state e, for each such r_i there exists at least one 2-valued lattice homomorphism h_i, with non-zero measure μ, such that $h_i(r_i) = 1$ (by TP). It follows that, for each $i = 1,..k$. either $r_i\wedge p\neq 0$ or $r_i\wedge p^\perp\neq 0$ (or else $h_i(p) = h_i(p^\perp) = 0$, contradicting TP).

Suppose, then, without loss of generality (as shown above), that $\dim(r_i)>2$. If $r_i\wedge p = r_i$, then $r_i\leq p$ and $e_{r_i}\leq p$. If $r_i\wedge p^\perp = r_i$, then $r_i\leq p^\perp$ and $e_{r_i}\leq p^\perp$. So, if either, $r_i\wedge p = r_i$ or $r_i\wedge p^\perp = r_i$, then either $e_{r_i}\leq p$ or $e_{r_i}\leq p^\perp$.

If $r_i\wedge p\neq r_i$ and $r_i\wedge p^\perp\neq r_i$, then either $0\neq r_i\wedge p<r_i$ or $0\neq r_i\wedge p^\perp<r_i$ (for either $r_i\wedge p\neq 0$ or $r_i\wedge p^\perp\neq 0$). Suppose $0\neq r_i\wedge p<r_i$ (a similar argument applies if $0\neq r_i\wedge p^\perp<r_i$). We write $r_i\wedge p = b$ for convenience. Clearly, $b\in D(e,R)$, since $r_i\in D(e,R)$ by R-PREF and $p\in D(e,R)$ by hypothesis, and TP applies. There are two cases to consider: (i) b is orthogonal to e_{r_i} in r_i, and (ii) b is not orthogonal to e_{r_i} in r_i. We first show that in either case we must have $e_{r_i}\in D(e,R)$.

(i) Suppose b is orthogonal to e_{r_i} in r_i, i.e. $b\leq e_{r_i}'$, where the $'$ here denotes the orthogonal complement in r_i. Consider all lattice automorphisms U that are rotations about e_{r_i} in r_i and the identity in $r_i^\perp$ [such rotations are possible because $\dim(r_i)>2$, by hypothesis]. Clearly, U preserves e and R, because U preserves the eigenspaces of R, and U preserves the projections of e onto the eigenspaces of r. Then, by DEF, $U(b)\in D(e,R)$, for all such rotations, and there are clearly sufficiently many rotations (regardless of b's dimension) to generate a set of elements $\{U(b)\}$ in $D(e,R)$ whose span is e_{r_i}'. It follows that $e_{r_i}'\in D(e,R)$, by lattice closure (i.e. by TP), and since $r_i\in D(e,R)$ by R-PREF, lattice closure further requires that $e_{r_i}\in D(e,R)$.

(ii) Suppose b is not orthogonal to e_{r_i} in r_i. We may suppose that b is not 1-dimensional [for if it is, we can instead consider b', the orthocomplement of b in r_i, since $\dim(r_i)>2$ by hypothesis]. Since b is skew to e_{r_i}, $b = c\vee d$, where $c\leq e_{r_i}'$ (i. e. c is orthogonal to e_{r_i} in r_i) and d is a ray skew to e_{r_i}—the projection of the ray e_{r_i} onto the subspace b. Consider a lattice automorphism U that is a 'reflection' through $e_{r_i}\vee c$ in r_i (thus U preserves e_{r_i} and c, but not d) and the identity in $r_i^\perp$. As in (i) above, U preserves e and R, so $U(b)\in D(e,R)$ and $b\wedge U(b) = c\in D(e,R)$ by lattice closure. As in (i), since c is orthogonal to e_{r_i} in r_i, we can further consider rotations of c about e_{r_i} in r_i to show that $e_{r_i}'\in D(e,R)$ and hence that $e_{r_i}\in D(e,R)$.

Now, since $e_{r_i}\in D(e,R)$ and $e_{r_i}\neq 0$, there exists a 2-valued homomorphism h_i with non-zero measure μ such that $h_i(e_{r_i}) = 1$. Suppose $p\in D(e,R)$, and $e_{r_i}\leq p$ and $e_{r_i}\leq p^\perp$. Then $e_{r_i}\wedge p = 0$ and $e_{r_i}\wedge p^\perp = 0$ (because e_{r_i} is a ray). It follows that for this homomorphism $h_i, h_i(P) = 0$ and $h_i(p^\perp) = 0$ contradicting TP. So either $e_{r_i} \leq p$ or $e_{r_i} \leq p^\perp$.

We conclude that if $p \in D(e,R)$, then either $e_{r_i} \leq p$ or $e_{r_i} \leq p^\perp$, for all $i = 1,\dots,k$. Maximality now requires that $D(e,R)$ contains the set of all such elements p, i.e. $D(e,R) = \{p: e_{r_i} \leq p \text{ or } e_{r_i} \leq p^\perp, i = 1,\dots k\} = L_{e_{r_1}e_{r_2}\dots e_{r_k}}$, because $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ satisfies TP, R-PREF, DEF and WEAK SEP.

To see that TP is satisfied, we need to show that $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ is closed under the ortholattice operations, that there exist 2-valued homomorphisms on $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$, and that the joint probabilities assigned by e to mutually compatible sets of elements $\{p_i\}_{i\in I}$ in $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ can be recovered as measures on a Kolmogorov probability space (X,F,μ), where X is the set of 2-valued homomorphisms on $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$.

We consider closure first. If $p \in L_{e_{r_1}e_{r_2}\dots e_{r_k}}$, then either $e_{r_i} \leq p$ or $e_{r_i} \leq p^\perp$, for all $i = 1,\dots,k$. So, $p^\perp \in L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ because either $e_{r_i} \leq p^\perp$ or $e_{r_i} \leq (p^\perp)^\perp = p$. To show that $p \wedge q \in L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ and $p \vee q \in L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ if $p \in L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ and $q \in L_{e_{r_1}e_{r_2}\dots e_{r_k}}$, we need to show that $e_{r_i} \leq p\wedge q$ or $e_{r_i} \leq (p\wedge q)^\perp$, and $e_{r_i} \leq p \vee q$ or $e_{r_i} \leq (p\vee q)^\perp$, for all $i = 1,\dots,k$. If $p \in L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ and $q \in L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ then, for each $i = 1,\dots,k$, $e_{r_i} \leq p$ or $e_{r_i} \leq p^\perp$, and $e_{r_i} \leq q$ or $e_{r_i} \leq q^\perp$. So, either (i) $e_{r_i} \leq p$ and $e_{r_i} \leq q$, in which case $e_{r_i} \leq p\wedge q$ and $e_{r_i} \leq p \leq p\vee q$, or (ii) $e_{r_i} \leq p$ and $e_{r_i} \leq q^\perp$ in which case $e_{r_i} \leq p \leq p \vee q$ and $e_{r_i} \leq q^\perp \leq p^\perp \vee q^\perp = (p\wedge q)^\perp$ or (iii) $e_{r_i} \leq p^\perp$ and $e_{r_i} \leq q$, in which case $e_{r_i} \leq q \leq p\vee q$ and $e_{r_i} \leq p^\perp \leq p^\perp \vee q^\perp = (p \wedge q)^\perp$, or (iv) $e_{r_i} \leq p^\perp$ and $e_{r_i} \leq q^\perp$, in which case $e_{r_i} \leq p^\perp \wedge q^\perp = (p\vee q)^\perp$ and $e_{r_i} \leq p^\perp \leq p^\perp \vee q^\perp = (p\wedge q)^\perp$.

To show the existence of 2-valued homomorphisms on $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$, consider k maps h_i: $L_{e_{r_1}e_{r_2}\dots e_{r_k}} \to \{0,1\}$ defined as follows: For any element $p \in L_{e_{r_1}e_{r_2}\dots e_{r_k}}$, $h_i(p) = 1$, if $e_{r_i} \leq p$ and $h_i(p) = 0$ if $e_{r_i} \leq p^\perp$. Each h_i is a 2-valued homomorphism on $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$. Clearly, $h_i(0) = 0$ and $h_i(I) = 1$, because $e_{r_i} \leq I = 0^\perp$. If $e_{r_i} \leq p$, $h_i(p) = 1$ and $h_i(p^\perp) = 0$, because $p = (p^\perp)^\perp$. Similarly, if $e_{r_i} \leq p^\perp$, then $h_i(p^\perp) = 1$ and $h_i(p) = 1$. So $h_i(p) = 1 - h_i(p^\perp)$. If $e_{r_i} \leq p$ and $e_{r_i} \leq q$, so that $h_i(p) = 1$ and $h_i(q) = 1$, then $e_{r_i} \leq p\wedge q$, so $h_i(p\wedge q) = 1$. If $e_{r_i} \leq p$ and $e_{r_i} \leq q^\perp$, so that $h_i(p) = 1$ and $h_i(q) = 0$, or $e_{r_i} \leq p^\perp$ and $e_{r_i} \leq q$, so that $h_i(p) = 0$ and $h_i(q) = 1$, then $e_{r_i} \leq p^\perp \vee q^\perp = (p\wedge q)^\perp$, so that $h_i(p\wedge q) = 0$. If $e_{r_i} \leq p^\perp$ and $e_{r_i} \leq q^\perp$, so that $h_i(p) = h_i(q) = 0$, then $e_{r_i} \leq p^\perp \vee q^\perp = (p\wedge q)^\perp$, so that $h_i(p\wedge q) = 0$. So $h_i(p\wedge q) = h_i(p)h_i(q)$. Since $p \vee q = (p^\perp \wedge q^\perp)^\perp$, it follows that $h_i(p\vee q) = h_i(p) \vee h_i(q) = h_i(p) + h_i(q) - h_i(p)h_i(q)$. So each h_i is a homomorphism.

[We note that the k 2-valued homomorphisms h_i on $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$ are in general the *only* 2-valued homomorphisms on $L_{e_{r_1}e_{r_2}\dots e_{r_k}}$. Since any 2-valued homomorphism that maps any ray in $(e_{r_1} \vee e_{r_2} \vee \dots \vee e_{r_k})^\perp$ onto 1 must map each of the rays e_{r_i}, $i = 1,\dots,k$, onto 0, it follows that any such homomorphism must map one of each orthogonal $(n-k)$-tuple of rays in $(e_{r_1} \vee e_{r_2} \vee \dots \vee e_{r_k})^\perp$ onto 1 and the remaining members of the $(n-k)$-tuple onto 0. The infimum of any two rays that are mapped onto 1 must also be mapped onto 1, but this is the zero element. So no 2-valued homomorphism can map any of the rays in

$(e_{r_1} \vee e_{r_2} \vee \ldots e_{r_k})^{\perp}$ onto 1—except, of course, when $\dim((e_{r_1} \vee e_{r_2} \vee \ldots \vee e_{r_k})^{\perp}) = 1$, i.e. when $n = k+1$. In all other cases ($n > k+1$), it follows that the only 2-valued homomorphisms on $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$ are the homomorphisms h_i, $i = 1, \ldots, k$, where each h_i maps the ray e_{r_i} onto 1 and every other ray in $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$ onto 0 (since all the other rays in $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$ are orthogonal to e_{r_i}).]

To generate the probabilities assigned by e to mutually compatible sets of elements $\{p_i\}_{i \in I}$, $p_i \in L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$ on the Kolmogorov probability space, the 2-valued homomorphism that maps e_{r_i} onto 1 (more precisely, the corresponding singleton subset) is assigned the measure $\mathrm{tr}(ee_{r_i})$, for $i = 1, \ldots, k$. (The homomorphism that maps $(e_{r_1} \vee e_{r_2} \vee \ldots \vee e_{r_k})^{\perp}$ onto 1, assuming $\dim((e_{r_1} \vee e_{r_2} \vee \ldots \vee e_{r_k})^{\perp}) = 1$, is assigned the measure 0.) To see that $\mu(\{h\colon h(p_1) = h(p_2) = \ldots = 1\}) = \mathrm{tr}(ep_1p_2 \ldots)$, first note that $p_1p_2 \ldots = p_1 \wedge p_2 \wedge \ldots$ for compatible projections p_i, $p_2 \ldots$, and so the product $p_1p_2 \ldots$ defines an element, call it p, that also belongs to $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$, since $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$ is closed under the ortholattice operations. Furthermore, $h(p_1) = h(p_2) = \ldots = 1$ if and only if $h(p_1 \wedge p_2 \wedge \ldots) = h(p) = 1$, since h is a lattice homomorphism. So it suffices to show that, for a general element $p \in L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$, $\mu(\{h{:}h(p) = 1\}) = \mathrm{tr}(ep)$.

Now any $p \in L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$ can be expressed as $p = e_{r_1} \vee e_{r_j} \vee \ldots \vee q$, for some $i, j, \ldots, q$, where q is the projection operator onto a subspace (possibly the zero subspace) orthogonal to *all* the e_{r_i}, i.e. $q \in (e_{r_1} \vee e_{r_2} \vee \ldots \vee e_{r_k})^{\perp}$. If $h(p) = 1$, $h(e_{r_1} \vee e_{r_j} \vee \ldots \wedge q) = h(e_{r_1}) \vee h(e_{r_j}) \vee \ldots \vee h(q) = 1$. So:

$$\mu(\{h\colon\ h(p) = 1\}) = \mu(\{h\colon\ h(e_{r_i}) \vee h(e_{r_j}) \vee \ldots \vee h(q) = 1\})$$

$$= \mu(\{h\colon\ h(e_{r_1}) = 1\}) + \mu(\{h\colon\ h(e_{r_j}) = 1\}) + \ldots,$$

because $\mu(\{h\colon\ h(q) = 1\}) = 0$ and $h(e_{r_1}) \neq h(e_{r_j})$ if $i \neq j$. It follows that:

$$\mu(\{h\colon\ h(p) = 1\}) = \mathrm{tr}(ee_{r_i}) + \mathrm{tr}(ee_{r_j}) + \ldots$$

$$= \mathrm{tr}(e(e_{r_i} + e_{r_j} + \ldots + q))$$

because $\mu(\{h\colon\ h(e_{r_i}) = 1\}) = \mathrm{tr}(ee_{r_i})$ and $\mathrm{tr}(eq) = 0$ and hence:

$$\mu(\{h\colon\ h(e_{r_i}) = 1\}) = \mathrm{tr}(e(e_{r_1} \vee e_{r_j} \vee \ldots \vee q))$$

$$= tr(ep)$$

since $e_{r_i}, e_{r_j}, \ldots$ and q are mutually orthogonal.

R-PREF is satisfied by construction. Since $e_{r_i} \in L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$, $i = 1, \ldots, k$, and every ray in $(e_{r_1} \vee e_{r_2} \vee \ldots \vee e_{r_k})^{\perp}$ belongs to $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$, it follows that every ray in e_{r_i}', $i = 1, \ldots, k$, (the orthocomplement of e_{r_i} in r_i) belongs to $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$, so $r_i \in L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$, $i = 1, \ldots, k$ by lattice closure (established above). The remaining $m-k$ elements r_i also belong to $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$, because each of these is represented by a subspace orthogonal to e and hence by a subspace in $(e_{r_1} \vee e_{r_2} \vee \ldots \vee e_{r_k})^{\perp}$. But since every ray in $(e_{r_1} \vee e_{r_2} \vee \ldots \vee e_{r_k})^{\perp}$ belongs to $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$, every subspace of $(e_{r_1} \vee e_{r_2} \vee \ldots \vee e_{r_k})^{\perp}$ belongs to $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$.

DEF is satisfied because lattice automorphisms that preserve e and R automatically preserve the non-zero projections of e onto the eigenspaces r_i of R, i.e. the rays e_{r_i}, $i = 1,\ldots,k$, that generate $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$.

To see that WEAK SEP is satisfied, consider a composite system, $S+S^*$, represented on a Hilbert space $H \otimes H^*$. Suppose the state of $S+S^*$ is $e \otimes e^*$ and the privileged observable of $S+S^*$ is an observable $R\&R^*$. Then, for all i, $e_{r_i} \leq p$ or $e_{r_i} \leq p^{\perp}$ if and only if, for all i, j, $e_{r_i} \otimes e^*_{r^*_j} \leq p \otimes I^*$ or $e_{r_i} \otimes e^*_{r^*_j} \leq p^{\perp} \otimes I^*$. It follows that $D(e \otimes e^*, R\&R^*) = D(e,R)$. QED

3. Interpretations

The above analysis shows that the determinate sublattices of the lattice of projection operators or subspaces, $L(H)$, of a Hilbert space H, representing the propositions ('yes–no' experiments) of a quantum mechanical system are just the lattices $D(e,R) = L_{e_{r_1} e_{r_2} \ldots e_{r_k}} = \{p\colon\ e_{r_i} \leq p \text{ or } e_{r_i} \leq p^{\perp},\ i = 1,\ldots,k\}$. These are the maximal subcollections of quantum propositions that can be determinately true or false, given the quantum state e and a preferred observable R, subject to certain constraints that essentially require these subcollections to be lattices determined by e and R on which sufficiently many truth valuations exist to recover the usual quantum statistics. The set of observables associated with $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$ includes any observable whose eigenspaces are spanned by rays in $L_{e_{r_1} e_{r_2} \ldots e_{r_k}}$. The set of maximal observables includes any maximal observable with k eigenvectors in the directions e_{r_i}, $i = 1,\ldots,k$.

The uniqueness theorem characterizes a class of 'no collapse' interpretations of quantum mechanics, where each interpretation involves the selection of a particular preferred determinate observable, and hence the selection, via the quantum state at a particular time, of a particular determinate sublattice with respect to which the probabilities defined by the quantum state have the usual epistemic significance. So the quantum probabilities defined on the determinate sublattice can be understood as measures of ignorance about the actual values of observables associated with propositions in the determinate sublattice, i.e. the actual values of the preferred determinate observable and observables that inherit determinate status via the quantum state and the preferred observable.

We note that Bell concluded his seminal critique of impossibility proofs for hidden variables in quantum mechanics[22] by observing that the equations of motion in Bohm's theory 'have in general a grossly non-local character', so that 'in this theory an explicit causal mechanism exists whereby the disposition of

[22]J. S. Bell, 'On the Problem of Hidden Variables in Quantum Mechanics', *op. cit.*, p. 452.

one piece of apparatus affects the results obtained with a distant piece'. He remarked that 'the Einstein–Podolsky–Rosen paradox is resolved in the way which Einstein would have liked least' and raised the question whether one could prove 'that *any* hidden variable account of quantum mechanics *must* have this extraordinary character'. Bell subsequently proved[23] that no hidden variable theory satisfying a locality constraint could reproduce the quantum statistics. Our theorem goes beyond this result by characterizing all possible 'completions' of quantum mechanics ('no collapse' interpretations), subject to certain natural constraints.

3.1. The Orthodox (Dirac–von Neumann) Interpretation

Without the projection postulate, the orthodox interpretation is a 'no collapse' interpretation of quantum mechanics in our sense. On the orthodox interpretation, an observable has a determinate value if and only if the state of the system is an eigenstate of the observable. Equivalently, the propositions that are determinately true or false of a system are the propositions represented by subspaces that either contain the state of the system, or are orthogonal to the state, i.e. the propositions assigned probability 1 or 0 by the state. The orthodox interpretation can therefore be formulated as the proposal that the preferred determinate observable is the unit observable I, and that $D(e,I) = L_e$ is the determinate sublattice of a system in the state e, where $L_e = \{p\colon\ e \leq p \text{ or } e \leq p^{\perp}\}$.

The choice of the preferred determinate observable as the unit observable leads to the measurement problem. For a composite system $S+M$ in an entangled state of the form $|e\rangle = \sum_i c_i|a_i\rangle|r_i\rangle$, neither R-propositions nor A-propositions belong to the sublattice L_e. In order to avoid the problem, Dirac and von Neumann assume the projection postulate,[24] that unitary evolution is suspended in the case of a measurement interaction, and the state of $S+M$ is projected onto the ray spanned by one of the unit vectors $|a_i\rangle|r_i\rangle$ with probability $|c_i|^2$, because only in such a state, according to the orthodox interpretation, do the observables A and R have determinate values.

3.2. Resolution of the Measurement Problem

There is nothing in the mathematical structure of quantum mechanics that forces the choice of the preferred determinate observable R as the unit observable I, and indeed there is every reason to avoid this choice because it leads to the measurement problem. 'No collapse' interpretations that seek to solve this problem represent alternative proposals for choosing R. For such interpretations, there is no measurement problem if R plays the role of a pointer observable in all measurement interactions, or an observable correlated with the pointer observable.

[23]J. S. Bell, 'On the Einstein–Podolsky–Rosen Paradox', *op. cit.*

[24]Dirac, *op. cit.*, p. 36; and von Neumann, *op. cit.*, pp. 351 and 418.

To see this, consider a model quantum mechanical universe consisting of two systems, S and M, associated with a Hilbert space $H_S \otimes H_M$. A measurement interaction between S and M, say a dynamical evolution of the quantum state of the composite system $S+M$ described by a unitary transformation that correlates eigenstates $|a_i\rangle$ of an S-observable A with eigenstates $|r_i\rangle$ of an M-observable R, will result in a state represented by a unit vector of the form $|e\rangle = \sum_i c_i |a_i\rangle |r_i\rangle$ (assuming initial pure states for S and M). If we take $I \otimes R$ as the preferred determinate observable, the projections of the ray e onto the eigenspaces $H_S \otimes r_i$ of $I \otimes R$ are the rays e_{r_i} spanned by the unit vectors $|a_i\rangle|r_i\rangle$. So for this state, the determinate sublattice contains propositions represented by the projection operators $a_i \otimes I_M$, where a_i here represents the projection operator onto the subspace in H_S spanned by the unit vector $|a_i\rangle$, i.e. propositions corresponding to the eigenvalues of A (and also, of course, propositions corresponding to pointer positions, represented by the projection operators $I_S \otimes r_i$).

Evidently, the same conclusion follows if we take some observable $I \otimes I \otimes T$ as the preferred observable, where R is correlated with T via the dynamical evolution of the quantum state in a measurement interaction, so that the state after the measurement takes the form $|e\rangle = \sum_i c_i |a_i\rangle |r_i\rangle |t_i\rangle$. In this case, the projections of the ray e onto the eigenspaces $H_S \otimes H_M \otimes t_i$ of $I \otimes T \otimes T$ are the rays e_{t_i} spanned by the unit vectors $|a_i\rangle|r_i\rangle|t_i\rangle$.

It follows that, *without introducing any measurement constraints on the determinate sublattices* (apart from the choice of R, which could still be justified on grounds *independent of measurement)*, we *derive* that the propositions corresponding to the observable correlated with an appropriate pointer observable in the 'entangled' state arising from a unitary transformation representing a quantum mechanical measurement interaction are determinately true or false. So we *derive* the interpretation of the probabilities defined by a quantum state for the eigenvalues of an observable A as 'the probabilities of finding the different possible eigenvalues of A in a measurement of A,' where a measurement is represented as a dynamical process that yields determinate values for A.

3.3 The Modal Interpretations of Kochen and of Dieks

The idea behind a 'modal' interpretation of quantum mechanics is that quantum states, unlike classical states, constrain possibilities rather than actualities—which leaves open the question of whether one can introduce 'value states'[25] that assign values to the observables of the theory, or equivalently, truth values to the corresponding propositions. As Van Fraassen puts it:[26]

[25]The terminology is Van Fraassen's. See B. van Fraassen, *Quantum Mechanics: An Empiricist View, op. cit.*, p. 275.
[26]B. van Fraassen, *ibid.*, p. 279.

In other words, the [quantum] state delimits what can and cannot occur, and how likely it is—it delimits possibility, impossibility, and probability of occurrence—but it does not say what actually occurs. The transition from the possible to the actual is not a transition *of* state, but a transition *described by* the state.

Apart from Van Fraassen's original version of the modal interpretation,[27] there are now a variety of other 'no collapse' interpretations of quantum mechanics that can be seen as modal in this sense, for example, the interpretations of Krips, Kochen, Healey, Dieks, and Bub.[28] All these modal interpretations share with Van Fraassen's interpretation the feature that an observable can have a determinate value even if the quantum state is not an eigenstate of the observable, so they preserve the linear, unitary dynamics for quantum states without requiring the projection postulate to validate the determinateness of pointer readings and measured observable values in quantum measurement processes.

The modal interpretations of Kochen and Healey exploit the polar decomposition theorem to define value states. By that theorem, any pure quantum state $|e\rangle$ of a system $S+S^*$ can be expressed in the form:

$$|e\rangle = \sum_i c_i |u_i\rangle \otimes |v_i\rangle$$

for some orthonormal set of vectors $\{|u_i\rangle\}$ in $H(S)$ and some orthonormal set $\{|v_i\rangle\}$ in $H(S^*)$. The decomposition is unique if and only if $|c_i|^2 \neq |c_j|^2$ for any $i \neq$j. In the non-degenerate case, the basic idea is to take the propositions that are determinately true or false for S in the quantum state $|e\rangle$ as the propositions represented by the Boolean algebra of projection operators generated by the set $\{P_{|u_i\rangle}\}$. (Similar remarks apply to S^*, of course.) There are alternative proposals for the degenerate case. Clifton's proposal,[29] closely related to Dieks', is to take the set of determinate propositions of S as the propositions represented by projection operators of the form P_1+P_2, where P_1 belongs to the spectral measure of the density operator W representing the reduced state of $|e\rangle$ for S, and P_2 belongs to the null space of W (i.e. $P_2 W = 0$). All S-propositions assigned probability 1 or 0 are included in this set.

[27]First formulated in B. van Fraaasen, 'Hidden Variables and the Modal Interpretation of Quantum Statistics', *op. cit.*

[28]S. Kochen, 'A New Interpretation of Quantum Mechanics', *op. cit.*; H. Krips, *The Metaphysics of Quantum Theory* (Oxford: Clarendon Press, 1987); R. Healey, *The Philosophy of Quantum Mechanics* (Cambridge: Cambridge University Press, 1988); D. Dieks, 'Quantum Mechanics Without the Projection Postulate and Its Realistic Interpretation', *op. cit.*; 'Modal Interpretation of Quantum Mechanics, Measurements, and Macroscopic Behavior', *op. cit.*; J. Bub, 'Quantum Mechanics Without the Projection Postulate', *op. cit.*; 'How to Interpret Quantum Mechanics', *op. cit.*

[29]R. Clifton, 'Independently Motivating the Kochen–Dieks Modal Interpretation of Quantum Mechanics', *British Journal for the Philosophy of Science* **46** (1995), 33–57.

The interpretations of Kochen and of Dieks can therefore be understood as the proposal that for any quantum state W of a system S, pure or mixed [where W arises from partial tracing over $H(S^*)$ if S is a subsystem of a system $S+S^*$], the determinate propositions of S are the propositions represented by the projection operators in the set: $\mathrm{Det}_{\mathrm{KD}}(S) = \{P\colon\ P = P_1+P_2, P_1 \in$ spectral measure of W, $P_2 \in$ null space of $W\}$.

As Clifton has shown,[30] an equivalent formulation is:

$$\mathrm{Det}_{\mathrm{KD}}(S) = \{P\colon\ PP_{W_i} \text{ or } 0, \forall P_{W_i} \in SR(W)\},$$

where $SR(W)$ is the set of projection operators in the spectral representation of W (i.e. the projection operators onto the eigenspaces corresponding to the non-zero eigenvalues of W).[31] It is then easy to see that $\mathrm{Det}_{\mathrm{KD}}(S)$ forms a sublattice of $H(S)$.

We can now prove a recovery theorem for the interpretations of Kochen and of Dieks, that these versions of the modal interpretation are 'no collapse' interpretations, in the sense of Theorem 1, for the preferred determinate observable $R = W \otimes I^*$, i.e. the determinate sublattices of S, as a subsystem of the system $S+S^*$, are the sublattices $D(e, W \otimes I^*)|_S$, where '$|_S$' denotes the restriction of the sublattice to the Hilbert space H of the subsystem S, e is the ray representing the quantum state of $S+S^*$, and W is the reduced state of e for the system S. (So the preferred determinate observable is not fixed, but changes with time as the state e evolves.)

Theorem 2 (Recovery Theorem)

If S is a subsystem of a quantum mechanical universe $S+S^*$ represented on a Hilbert space $H \otimes H^*$, and the state of $S+S^*$ is represented by a ray $\in H \otimes H^*$, then

$$D(e, W \otimes I^*)|_S = \mathrm{Det}_{\mathrm{KD}}(S),$$

Proof: We introduce the following notation to capture the distinctions required to formulate the proof precisely:

W_i: the ith eigenspace of W (in H).
P_{W_i}: the projection operator onto W_i (in H).
$(W \otimes I^*)_i$: the ith eigenspace of $W \otimes I^*$ (in $H \otimes H^*$).
$P_{(W \otimes I^*)_i}$: the projection operator onto $(W \otimes I^*)_i$ (in $H \otimes H^*$).
$e_{(W \otimes I^*)_i}$: the non-zero projection of the ray e onto $(W \otimes I^*)_i$ (in $H \otimes H^*$).
$P_{e_{(W \otimes I^*)_i}}$: the (non-zero) projection operator onto the ray $e_{(W \otimes I^*)_i}$ (in $H \otimes H^*$).

[30]R. Clifton, 'Independently Motivating the Kochen–Dieks Modal Interpretation of Quantum Mechanics', *op. cit.*

[31]The spectral measure of W is the Boolean algebra generated by $SR(W)$.

Note that there is a 1–1 correspondence between the eigenspaces W_i in H and $(W\otimes I^*)_i$ in $H\otimes H^*$, because w_i is an eigenvalue of W if and only if w_i is an eigenvalue of $W\otimes I^*$.

In our formulation of the uniqueness theorem in Section 2, we showed that

$$D(e,R) = L_{e_{r_1}e_{r_2}\ldots e_{r_k}} = \{p\colon\ e_{r_i}\leq p \ \text{ or } \ e_{r_i}\leq p^{\perp},\ i=1,\ldots,k\},$$

where the e_{r_i} are the non-zero projections of e onto the k eigenspaces of R. Writing P for the projection operator corresponding to the lattice element p, and $P_{e_{r_i}}$ for the projection operator corresponding to the lattice element e_{r_i}, the determinate sublattice of projection operators can be expressed equivalently as:

$$\{P\colon\ PP_{e_{r_i}} = P_{e_{r_i}} \ \text{ or } 0,\ \forall i\},$$

where the quantifier over the index i ranges over all non-zero projections of e onto the eigenspaces of R.

So what we want to show is that

$$\begin{aligned} D(e,W\otimes I^*)|_S &\equiv \{P\colon (P\otimes I^*)P_{e_{(W\otimes I^*)_i}} = P_{e_{(W\otimes I^*)_i}} \ \text{ or } 0,\ \forall i\} \\ &= \mathrm{Det}_{\mathrm{KD}}(S) \\ &\equiv \{P\colon\ PP_{W_i} = P_{W_i} \ \text{ or } 0,\ \forall i\}. \end{aligned}$$

The quantifier over the index i in the expression for $\mathrm{Det}_{\mathrm{KD}}(S)$ ranges over all projection operators P_{W_i} in $SR(W)$. The quantifier over the index i in the expression for $D(e,W\otimes I^*)|_S$ ranges over all non-zero projections of e onto the eigenspaces of $W\otimes I^*$, including in principle the eigenspace corresponding to the zero eigenvalue (the null space of $W\otimes I^*$), if $W\otimes I^*$ has a zero eigenvalue. Since W is the reduced state of e for the system S, e lies entirely in the span of the non-zero eigenvalue eigenspaces of $W\otimes I^*$, and hence has zero projection onto the zero eigenvalue eigenspace, if such an eigenspace exists. So the quantifier in the expression for $D(e,W\otimes I^*)|_S$ in fact ranges over the eigenspaces with projection operators in the set $SR(W\otimes I^*)$, which are in 1–1 correspondence with projection operators in the set $SR(W)$. In the following, quantification over the index i refers to a set of projection operators or subspaces in 1–1 correspondence with the set $SR(W)$.

(i) We first prove that

$$\mathrm{Det}_{\mathrm{KD}}(S)\subseteq D(e,W\otimes I^*)|_S.$$

It suffices to show that the generators of $\mathrm{Det}_{\mathrm{KD}}(S)$ are in $D(e, W\otimes I^{*})|_{S}$, because $D(e, W\otimes I^{*})|_{S}$ is an ortholattice and so must include the lattice $\mathrm{Det}_{\mathrm{KD}}(S)$ as a sublattice if it includes the generators of $\mathrm{Det}_{\mathrm{KD}}(S)$.

The generators of $\mathrm{Det}_{\mathrm{KD}}(S)$ are the projection operators in the set:

$$\{P_{W_i}\colon\ \forall i\}\cup\{P\colon\ PP_{W_i}=0,\ \forall i\}.$$

To see that $\{P_{W_i}\colon\ \forall i\}\subseteq D(\mathrm{e}, W\otimes I^{*})|_{S}$, note that

$$P_{(W\otimes I^{*})_i}\in D(e, W\otimes I^{*}),\ \forall i,$$

because these are just the projection operators onto the non-zero eigenvalue eigenspaces of the preferred determinate observable $W\otimes I^{*}$, which always belong to $D(e, W\otimes I^{*})$ by assumption. But since

$$P_{(W\otimes I^{*})_i}=P_{W_i}\otimes I^{*},\ \forall i,$$

it follows that

$$P_{W_i}\in D(e, W\otimes I^{*})|_{S},\ \forall i.$$

To see that $\{P\colon\ PP_{W_i}=0,\ \forall i\}\subseteq D(e, W\otimes I^{*})|_{S}$, note that if $PP_{W_i}=0$ for all $P_{W_i}\in SR(W)$, then

$$(P\otimes I^{*})(P_{W_i}\otimes I^{*})=(P\otimes I^{*})P_{(W\otimes I^{*})_i}=0,\ \forall i,$$

and so

$$(P\otimes I^{*})P_{e_{(W\otimes I^{*})_i}}=0,\ \forall i.$$

But if $(P\otimes I^{*})P_{e_{(W\otimes I^{*})_i}}=0$ for all non-zero projections of the ray e onto the non-zero eigenvalue eigenspaces of $W\otimes I^{*}$ then, since e has zero projection onto the zero eigenvalue eigenspace of $W\otimes I^{*}$ (if such an eigenspace exists), $(P\otimes I^{*})P_{e_{(W\otimes I^{*})_i}}=0$ for all non-zero projections of e onto all the eigenspaces of $W\otimes I^{*}$, and so, by definition:

$$P\in D(e, W\otimes I^{*})|_{S}.$$

(ii) We now prove that

$$D(e, W\otimes I^{*})|_{S}\subseteq \mathrm{Det}_{\mathrm{KD}}(S).$$

Suppose the state vector $|e\rangle$ can be represented in *one* of its decompositions with respect to $H \otimes H^*$ as:

$$|e\rangle = \sum_{ij} c_{ij} |u_{ij}\rangle |v_{ij}\rangle.$$

We can express this vector as

$$|e\rangle = \sum_i \left(\sum_j c_{ij} |u_{ij}\rangle |v_{ij}\rangle\right),$$

where for fixed $i, |c_{ij_1}|^2 = |c_{ij_2}|^2 = \ldots \equiv |c_i|^2$ and the index i ranges over the *distinct* numbers $|c_i|^2$. These are just the non-zero eigenvalues of W that index $SR(W)$, and so:

$$P_{W_i} = \sum_j |u_{ij}\rangle\langle u_{ij}|.$$

We now have, for all i:

$$|e_{(W\otimes I^*)_i}\rangle = P_{(W\otimes I^*)_i}|e\rangle = P_{W_i} \otimes I^* |e\rangle = \sum_j c_{ij} |u_{ij}\rangle |v_{ij}\rangle.$$

Suppose $P \in D(e, W \otimes I^*)_S$, then:

$$(P \otimes I^*) P_{e_{(W\otimes I^*)_i}} = P_{e_{(W\otimes I^*)_i}} \text{ or } 0, \ \forall i.$$

It follows that

$$(P \otimes I^*) |e_{(W\otimes I^*)_i}\rangle = |e_{(W\otimes I^*)_i}\rangle \text{ or } |0\rangle, \ \forall i,$$

and so

$$\sum_j c_{ij} P |u_{ij}\rangle |v_{ij}\rangle = \sum_j c_{ij} |u_{ij}\rangle |v_{ij}\rangle \text{ or } |0\rangle, \ \forall i.$$

For a fixed i, this implies that either $P|u_{ij}\rangle = |u_{ij}\rangle$ for all j, or $P|u_{ij}\rangle = |0\rangle$ for all j, because $c_{ij} \neq 0$ for any j. And since $\{|u_{ij}\rangle, \ \forall j\}$ is a basis for the eigenspace W_i:

$$PP_{W_i} = P_{W_i} \text{ or } 0, \ \forall i,$$

i.e.

$$P \in \mathrm{Det}_{\mathrm{KD}}(S).$$

3.4. Bohmian Mechanics

Bohm's 1952 hidden variable theory or 'causal' interpretation[32] can be understood as a proposal for implementing an interpretation in which the preferred determinate observable R is fixed once and for all as position in configuration space, instead of being defined by the time evolving quantum state as in the modal interpretations of Kochen and Dieks. (Alternative versions of Bohm's theory taking R as momentum or some other observable instead of position are considered by Epstein[33] and Stone)[34].

Bohmian dynamics arises as the dynamics of 'value states' on the determinate sublattice $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$ (i.e. states defined by 2-valued homomorphisms on $L_{e_{r_1}e_{r_2}\ldots e_{r_k}}$) as the quantum state e evolves in time. As we have seen, value states defined by 2-valued homomorphisms (with non-zero measure) are in 1–1 correspondence with the rays e_{r_i}, hence with the eigenspaces r_i of R, and assign the same value to e_{r_i} and r_i, for all i. Since the evolution of such states is completely determined by the evolution of e and of R, we want an equation of motion for the determinate values of R that will preserve the distribution of R-values specified by e, as e evolves in time according to Schrödinger's time-dependent equation of motion. (Several authors have proposed dynamical evolution laws for value states in the modal interpretations of Kochen and of Dieks, where the preferred determinate observable is not fixed but is defined by the quantum state and so changes with time as the state evolves.)[35]

Recall that Bohm extracts two real equations from Schrödinger's time-dependent complex equation of motion for the wave function of a single particle of mass M,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi + V\psi,$$

by substituting $\psi = \mathrm{R}\exp\left(\frac{iS}{\hbar}\right)$:[36]

[32]D. Bohm, 'A Suggested Interpretation of Quantum Theory in Terms of 'Hidden Variables': Parts I and II', *op. cit.*

[33]T. Epstein, 'The Causal Interpretation of Quantum Mechanics', *Physical Review* **89** (1952), 319 and *Physical Review* **91** (1953), 985.

[34]A. Stone, 'Does the Bohm Theory Solve the Measurement Problem?', *Philosophy of Science* **61** (1994), 250–266.

[35]D. Dieks, 'Modal Interpretations of Quantum Mechanics, Measurements, and Macroscopic Behavior' *op. cit.*; P.E. Vermaas, 'Unique Transition Probabilities in the Modal Interpretation', *Studies in History and Philosophy of Modern Physics* **27** (1996) 133–159; G. Bacciagaluppi and M. Dickson, 'Modal Interpretations with Dynamics', manuscript.

[36]The symbol 'R' here should not, of course, be confused with the symbol 'R' for the preferred determinate observable.

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2M} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0,$$

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(\frac{R^2 \nabla S}{M}\right) = 0.$$

The first equation (derived from the real part of the Schrödinger equation) can be interpreted as a Hamilton–Jacobi equation for the motion of the particle under the influence of a potential function V and an additional 'quantum potential' $\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$. The trajectories of these particles are given by the solutions to the equation:

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{J}}{\rho} = \frac{\nabla S}{M} = \frac{\hbar}{M}\mathrm{Im}\left(\frac{\nabla\psi}{\psi}\right) = \frac{\hbar}{2iM}\frac{\psi^*\nabla\psi - \psi\nabla\psi^*}{|\psi|^2},$$

where Im( ) denotes the imaginary part of ( ), $\rho = R^2 = |\psi|^2$, and $J = \frac{R^2\nabla S}{M} = \frac{\hbar}{M}\mathrm{Im}(\psi^*\nabla\psi)$, i.e. the particle trajectories are given by the integral curves of a velocity field defined by the gradient of the phase S. So the trajectories $\mathbf{x}(t)$ depend on the wave function ψ. The second equation (derived from the imaginary part of the Schrödinger equation) can be written as a continuity equation for an ensemble density $\rho = |\psi|^2$, and a probability current $\mathbf{J}$:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0.$$

The continuity equation guarantees that if $\rho = |\psi|^2$ initially, ρ will remain equal to $|\psi|^2$ at all times.

Vink[37] has shown how to formulate a dynamics for any always determinate observable by generalizing a proposal by Bell[38] for constructing stochastic Bohm-type trajectories for fermion number density regarded as an always determinate observable or 'beable' for quantum field theory.

Vink considers an arbitrary complete set of commuting observables R^i $(i = 1,2,\ldots,I)$, with simultaneous eigenvectors $|r^1_{n^1}, r^2_{n^2},\ldots,r^I_{n^I}\rangle$, where the $n^i = 1,2,\ldots,N^i$ label the finite and discrete eigenvalues of R^i. Suppressing the index i, these are written as $|r_n\rangle$. (Equivalently, one can take the $|r_n\rangle$ to be

[37] J. C. Vink, 'Quantum Mechanics in Terms of Discrete Beables', *Physical Review* **A48** (1993), 1808–1818.

[38] J. S. Bell, 'Beables for Quantum Field Theory', in *Speakable and Unspeakable in Quantum Mechanics, op. cit.*, pp. 173–180. See pp. 176–177.

the different eigenvectors of a maximal observable R of which each of the R^i is a function. Then suppling the dynamics for R automatically induces a dynamics on each R^i.) The time evolution of the state vector is given by the equation of motion:

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle,$$

or

$$i\hbar\frac{d\langle r_n|\psi\rangle}{dt} = \langle r_n|H\psi\rangle = \sum_m \langle r_n|H|r_m\rangle\langle r_m|\psi\rangle$$

in the R-representation.

The imaginary part of this equation yields the continuity equation:

$$\frac{dP_n}{dt} = \frac{1}{\hbar}\sum_m J_{nm},$$

where the probability density P_n and the current matrix J_{nm} are defined by:

$$P_n(t) = |\langle r_n|\psi(t)\rangle|^2$$

$$J_{nm}(t) = 2\mathrm{Im}(\langle\psi(t)|r_n\rangle\langle r_n|H|r_m\rangle\langle r_m|\psi(t)\rangle)$$

For the non-maximal (degenerate) observables R^i, the probability density and current matrices are defined by summing over the remaining indices, e.g.

$$P^i_n = \sum_r |\langle r^i_{n^i}, r|\psi\rangle|^2,$$

where $r = r^j_{m^j}$ for $j \neq i$, and similarly for J^i_{nm}.

We want a stochastic dynamics for the discrete observable R consistent with the continuity equation. Suppose the jumps in R-values are governed by transition probabilities $T_{mn}dt$, where $T_{mn}dt$ denotes the probability of a jump from value r_n to value r_m in time dt.

The transition matrix gives rise to time-dependent probability distributions of R-values. It follows that the rate of change of the probability density $P_n(t)$ for r_n must satisfy the master equation:

$$\frac{dP_n(t)}{dt} = \sum_m (T_{nm}P_m - T_{mn}P_n).$$

But, from the Schrödinger equation, $dP_n(t)/dt$ also satisfies the continuity equation

$$\frac{dP_n(t)}{dt} = \frac{1}{\hbar}\sum_m J_{nm},$$

so we require

$$\frac{J_{nm}}{\hbar} = T_{nm}P_m - T_{mn}P_n.$$

We want solutions for the transition matrix T, given P and J, with $T_{mn} \geq 0$. Since $J_{mn} = -J_{nm}$ (hence $J_{nn} = 0$), the above equation yields $n(n-1)/2$ equations for the n^2 elements of T. So there are many solutions. Bell's choice[39] was:

$$T_{nm} = \frac{J_{nm}}{\hbar P_m}, \text{ if } J_{nm} \geq 0$$

$$T_{nm} = 0, \text{ if } J_{nm} \leq 0$$

For $n = m$, T_{nn} is fixed by the normalization $\sum_m T_{mn} dt = 1$.

Vink shows that Bell's solution for the transition matrix leads to Bohm's theory in the continuum limit, when R is position in configuration space. For example, consider a single particle of mass M on a 1-dimensional lattice. Let $x = nd$, with $n = 1,2,\ldots,N$ and d the lattice distance. Vink shows that to first order in d:

$$J_{nm} = \frac{\hbar}{Md}\,[S'(md)P_m\delta_{m,n-1} - S'(md)P_m\delta_{m,n+1}]$$

where S is the phase, and the derivative, F', of a function F on the lattice is defined as $F'(x) = \dfrac{F(x+d) - F(x)}{d}$. Thus Bell's solution yields:

$$T_{nm} = \frac{S'(md)}{Md}\delta_{m,n-1}, \; S'(md) \geq 0$$

$$T_{nm} = \frac{S'(md)}{Md}\delta_{m,n+1}, \; S'(md) \leq 0$$

For positive $S'(md)$ the particle can jump from site m to site $m+1$ with probability $\dfrac{|S'(md)|dt}{Md}$, and for negative $S'(md)$ the particle can jump from site

[39]In J. S. Bell, 'Beables for Quantum Field Theory', *op. cit.*

m to m -1 with the same probability. So the nearest neighbour interactions in the Hamiltonian induce jumps only between neighbouring lattice sites for this transition matrix. Since each jump is over a distance d, the average displacement in a time interval dt is:

$$\mathrm{d}x = \frac{\mathrm{S}'(x)\mathrm{d}t}{M},$$

i.e.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{S}'(x)}{M}.$$

As $d \rightarrow 0$, $\mathrm{S}' \rightarrow \partial_x \mathrm{S}$, and so in the continuum limit:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial_x \mathrm{S}}{M},$$

as for the continuous trajectories in Bohm's theory.

Vink shows that the dispersion vanishes in the limit as $d \rightarrow 0$, and so the trajectories become smooth and identical to the trajectories in Bohm's theory as $d \rightarrow 0$. Other solutions for the transition matrix induce jumps between distant sites, so differentiable deterministic trajectories are not recovered even in the continuum limit. Nelson's stochastic dynamics[40] is characterized by one such solution. Bohmian dynamics, as the continuum limit of a stochastic dynamics, appears to be the unique deterministic dynamics for a 'no collapse' interpretation of quantum mechanics based on position in configuration space as an always determinate observable.

Vink himself proposes to take *all* observables as simultaneously determinate, with their determinate values all evolving independently in accordance with the above stochastic dynamics. These determinate values do not satisfy the Kochen and Specker functional relationship constraint in general, but it turns out that this constraint is satisfied for the measured values of simultaneously measured observables. This follows because a measurement, represented as an evolution of the quantum state to a linear superposition of product eigenstates of a pointer observable and measured observable, leads to an effective collapse of the state to one of the product eigenstates, as in the analysis of measurement in Bohm's theory discussed below. The determinate values of observables in eigenstates of these observables are just the corresponding eigenvalues, and the functional relationship constraint is satisfied for common eigenstates of a set of observables.[41]

[40]E. Nelson, 'Derivation of the Schrödinger Equation from Newtonian Mechanics', *Physical Review* **150** (1966), 1079–1085.

[41]For a discussion, see J. Bub, 'Interference, Noncommutativity, and Determinateness in Quantum Mechanics', *Topoi* **14** (1995), 39–43.

There are two empirically equivalent proposals in the literature for handling observables other than position in configuration space in Bohm's theory. Bohm, Schiller, and Tiomno,[42] and also Dewdney,[43] Holland and Vigier,[44] and Bohm and Hiley[45] treat spin as well as position in configuration space as an always determinate observable. So a quantum particle always has a determinate spin property as well as a determinate position in configuration space. The spin observable can take determinate values that are not eigenvalues of spin in states that are not eigenstates of spin, but the measured value of spin is always an eigenvalue of spin. This treatment of spin could, in principle, be extended to all observables.

The alternative way of handling observables other than position in configuration, space, favoured by authors such as Bell,[46] Dürr, Goldstein and Zanghì,[47] Albert,[48] and Cushing,[49] conforms to an interpretation in our sense. The propositions in a Bohmian determinate sublattice are selected by the quantum state and position in configuration space as the only preferred determinate observable R. While position in configuration space is always determinate, other observables are sometimes determinate and sometimes indeterminate, depending on the quantum state. These other observables, with temporary determinate status inherited from R and the quantum state, can be associated with dispositions of the system. The observable spin, for example, is determinate in some quantum states and indeterminate in other quantum states. The possibility that the state can evolve to a form in which a spin component has a determinate value reflects a feature of the dynamics of the quantum state understood as a new kind of field in R-space, that this field evolves in a certain way in the presence of magnetic fields, i.e. it reflects a disposition of the system to undergo a certain kind of change of R-value under certain physical conditions. The different eigenvalues of the spin component mark the different possible changes in R associated with this evolution of the state. As we have seen, when the state takes a form in which a spin component has a determinate value (where this value is to be understood dispositionally), the probabilities

[42]D. Bohm, R. Schiller and J. Tiomno, 'A Causal Interpretation of the Pauli Equation', *Suppl. Nuovo Cimento* **1** (1955), 48–66.

[43]C. Dewdney, 'Constraints on Quantum Hidden-Variables and the Bohm Theory', *Journal of Physics A: Math. Gen.* **25** (1992), 3615–3626.

[44]P. R. Holland and J. P. Vigier, 'The Quantum Potential and Signalling in the Einstein–Podolsky–Rosen Experiment', *Foundations of Physics* **18** (1988), 741–749.

[45]D. Bohm and B. J. Hiley, *The Undivided Universe: An Ontological Interpretation of Quantum Theory* (London: Routledge, 1993).

[46]J. S. Bell, 'Quantum Mechanics for Cosmologists'; and 'On the Impossible Pilot Wave', in *Speakable and Unspeakable in Quantum Mechanics, op. cit.*

[47]D. Dürr, S. Goldstein and N. Zanghì, 'Quantum Equilibrium and the Origin of Absolute Uncertainty', *Journal of Statistical Physics* **67** (1992), 843–907.

[48]D. Albert, *Quantum Mechanics and Experience* (Cambridge, MA: Harvard University Press, 1992).

[49]J. T. Cushing, *Quantum Mechanics: Historical Contingency and the Copenhagen Hegemony* (Chicago: University of Chicago Press, 1994).

assigned to the different eigenvalues of the spin component can be interpreted epistemically, as measures over a range of possible spin properties, one of which is actual. But these spin properties play no role in the dynamical evolution of the R-trajectories, which depends entirely on the initial value of R and the quantum state. From this point of view, the only real change in a Bohmian universe is the change in the quantum state and the change in R, and this suffices to account for all quantum phenomena. The propositions in the determinate sublattice for a given quantum state are the propositions that can be taken as determinately true or false consistently with R-propositions, and these will be associated with observables that we can *interpret* as measured via R as the pointer observable when the quantum state takes an appropriate form correlating values of R with values of these observables. A 'measurement' in this sense reveals dispositions of the system, grounded in the value of R and the quantum state, not pre-existing real occurrent values of any measured observable. So there is no interpretative advantage in taking any observables other than R as always determinate.

To illustrate, we consider the measurement of spin-related observables on Bohm's theory (following an analysis by Pagonis and Clifton).[50]

Let S_x^2, $S_{x'}^2$, S_y^2, $S_{y'}^2$, S_z^2 represent the squared components of spin in the x, x', y, y' and z directions of a spin-1 particle, respectively, where x, y, z and x', y', z form two orthogonal triples of directions with the z-direction in common. Each of these observables has eigenvalues 1 and 0 (taking units in which $\hbar = 1$ and a spin component has eigenvalues -1, 0 and $+1$), corresponding respectively to a plane and a ray in H_3. The three 0-eigenrays of S_x^2, S_y^2, S_z^2 form an orthogonal triple in H_3, and the three 0-eigenrays of $S_{x'}^2$, $S_{y'}^2$, S_z^2 form another orthogonal triple in H_3, with the 0-eigenray of S_z^2 in common.

Define the observables H and H' as:

$$H = S_x^2 - S_y^2,$$

$$H' = S_{x'}^2 - S_{y'}^2.$$

The observables H and H' are maximal, with three eigenvalues, -1, 0 and $+1$, and incompatible (i.e. the corresponding operators do not commute). The eigenvalues -1, 0 and $+1$ of H correspond to the orthogonal triple of eigenrays defined by the 0-eigenrays of S_x^2, S_y^2 and S_z^2, respectively. The eigenvalues -1, 0 and $+1$ of H' correspond to the orthogonal triple of eigenrays defined by the 0-eigenrays of $S_{x'}^2$, S_z^2 and $S_{y'}^2$, respectively.

The non-maximal observable S_z^2 can be represented as a function of H and also as a function of H':

[50]C. Pagonis and R. Clifton, 'Unremarkable Contextualism: Dispositions in the Bohm Theory', *Foundations of Physics* **25** (1995), 281–296.

$$S_x^2 = H^2$$
$$= H'^2.$$

So S_z^2 can be measured via a measurement of H or of H'.

The observable H can be measured, in principle, by passing the particle through a suitable inhomogeneous electromagnetic field, which functions much like a Stern–Gerlach magnet for the measurement of spin (see Swift and Wright[51]). The interaction of the particle with the field will be governed by a Hamiltonian of the form:

$$H_{\text{int}} = gi^{-1}\frac{\partial}{\partial q}H,$$

where g is a positive coupling constant that is non-zero only during the interaction, and q is a component of the particle's position. We make the usual assumption that the measurement is impulsive, so that the Schrödinger equation reduces to

$$\frac{\partial\psi}{\partial t} = -iH_{\text{int}}\psi$$

during the interaction, and has the solution

$$\psi(q,t) = \exp(-g\frac{\partial}{\partial q}Ht)\psi(q,0).$$

Taking the initial quantum state as

$$\psi(q,0) = \varphi(q)\sum_j c_j|H = j\rangle,$$

where $\varphi(q)$ is a narrow wave packet symmetric about $q = 0$, and $|H = j\rangle$ is an eigenstate of H, the state at any time t during the interaction is:

$$\psi(q,t) = \sum_j c_j\varphi(q-gjt)|H = j\rangle.$$

With a suitable choice for the coupling constant g, after a time $t \geq T$ (at the end of the interaction), gt will be significantly larger than the width of the packet $\varphi(q)$, so that the overlap between adjacent wave packets $\varphi(q-gjt)$ is negligible. As a result of the interaction, the particle's H-value will therefore become correlated with the particle's q-position. The Bohmian particle

[51] A. R. Swift and R. Wright, 'Generalized Stern–Gerlach Experiments and the Observability of Arbitrary Spin Operators', *Journal of Mathematical Physics* **21** (1980), 77–82.

trajectories during the interaction are governed by the equation of motion for q:

$$\frac{dq}{dt} = \frac{J_q}{\rho},$$

where $\rho(q) = |\psi(q,t)|^2$ is the probability density and $J_q = \psi^*(q,t)gH\psi(q,t)$ is the q-component of the probability current. So

$$\frac{dq}{dt} = \frac{g\sum_j j|c_j|^2|\varphi(q-gjt)|^2}{\sum_j |c_j|^2|\varphi(q-gjt)|^2}.$$

This equation can be solved to yield different trajectories for the different initial positions of the particle in the initial wave packet. By the continuity equation, $\frac{\partial \rho}{\partial t} + \frac{\partial J_q}{\partial q} = 0$, the particle trajectories at time $t \geq T$ will be distributed over the positions $q = gjt$ with probabilities $|c_j|^2$, *for* $j = -1,0,+1$. So q acts as a measurement pointer for H-values. A displacement of the particle from its initial position in the narrow wave packet $\varphi(q)$ centred about $q = 0$ by an amount gjt as the particle leaves the field can be understood as a measurement of H with the outcome $H = j$.

Now, the same analysis for H' instead of H will yield an equation of motion for the particle trajectories in terms of a position coordinate q':

$$\frac{dq'}{dt} = \frac{Jq'}{\rho}$$
$$= \frac{g\sum_j j|c_j'|^2|\varphi(q'-gjt)|^2}{\sum_j |c_j'|^2|\varphi(q'-gjt)|^2},$$

where the initial state of the particle is

$$\psi(q',0) = \varphi(q')\sum_j c_j'|H' = j\rangle,$$

and q' is the position coordinate of the particle that becomes correlated with the value of H' during the interaction (in which the particle is passed through an electromagnetic field oriented in a direction suitable for a measurement of $H' = S^2_{x'} - S^2_{y'}$ rather than $H = S^2_x - S^2_y$). We assume that the form of the initial position wave function is the same for q' as for q, and that the strength, g, and duration, T, of the interaction is the same for H' as for H.

Distinct possible particle trajectories cannot cross along the q-axis in an H-measurement, or the q'-axis in an H'-measurement, because the equation of motion is deterministic and gives velocity only as a function of position. This means that in an H-measurement, trajectories that end up, after time $t \geq T$, at one of the three possible final q-positions, $-gt$, 0 or gt, in that order, begin in one of three possible q-regions in the initial wave packet $\varphi(q)$, ordered from negative to positive values of q. The relative sizes of the q-regions in an H-measurement will differ from the relative sizes of the q'-regions in an H'-measurement, because fractions $|c_{-1}|^2, |c_0|^2, |c_{+1}|^2$ of the initial q-positions (in an ensemble distributed according to $|\varphi(q)|^2$ end up at the positions $q = -gt$, $q = 0$, $q = gt$, respectively (corresponding to the results $H = -1$, 0, +1 in an H-measurement), while *different* fractions $|c_{-1}'|^2, |c_0'|^2, |c_{+1}'|^2$ of the initial positions (in an ensemble distributed according to the same distribution function $|\varphi(q')|^2$) end up at the positions $q' = -gt$, $q' = 0$, $q' = gt$, respectively (corresponding to the results $H' = -1$, 0, +1 in an H'-measurement). Note that $|c_i|^2 \neq |c_i'|^2$, unless $H = H'$. It follows that a given initial position of the particle will be affected differently by different interaction Hamiltonians, and so a measurement of S_z^2 via an H-measurement need not yield the same result as a measurement of S_z^2 via an H'-measurement, for the same initial position and quantum state of the particle.

In this sense, Bohm's theory violates the functional relationship constraint. If values are assigned to all observables of a spin-1 particle as the values that *would be* obtained on measurement (where a measurement of an observable is understood as an evolution of the quantum state of the particle to a form that correlates position values with values of the observable in the above sense) then, for some initial positions of the particle, the value assigned to S_z^2 will *not* be equal to the square of the value assigned to H^2, and *also* equal to the square of the value assigned to H'^2. For example, suppose the initial position of the particle is such that a measurement of H would yield the value +1, while a measurement of H' would yield the value 0. If these values are assigned to H and H', then the functional relationship constraint requires that $S_z^2 = 1$ and also that $S_z^2 = 0$, since $S_z^2 = H^2 = H'^2$. Of course, this contradiction does not show any inconsistency in Bohm's interpretation of quantum mechanics. Rather, it shows that observables like H, H' and S_z—observables associated with dispositions of the system—are 'contextual', in the sense that no determinate values can be attributed to them, except in the context of a specific measurement process, understood as the dynamical evolution of the quantum state to a certain form.

In a similar sense, Bohm's theory is non-local. Consider two spin-1/2 particles, S_1 and S_2, in the singlet spin state at time $t = 0$:

$$\psi(q_1,q_2,0) = \varphi(q_1)\varphi(q_2)\left(\frac{1}{\sqrt{2}}|+\rangle_1|-\rangle_2 - \frac{1}{\sqrt{2}}|-\rangle_1|+\rangle_2\right)$$

as in Bohm's version of the Einstein–Podolsky–Rosen experiment.[52] A measurement of spin in the z-direction on S_1 (via a Stern–Gerlach interaction at S_1 that correlates the position of S_1 to the z-spin of S_1) induces an evolution of the quantum state of the 2-particle system to a form:

$$\psi(q_1,q_2,t) = \varphi(q_2)\left[\frac{1}{\sqrt{2}}\varphi(q_1+gt)|+\rangle_1|-\rangle_2 - \frac{1}{\sqrt{2}}\varphi(q_1-gt)|-\rangle_1|+\rangle_2\right]$$

where the wave packets $\varphi(q_1-gt)$ and $\varphi(q_1+gt)$ for the relevant position coordinates of S_1 are separated with negligible overlap. The q_1 position coordinate of the particle S_1 is effectively associated with just one of these wave packets. This means that the position of the 2-particle system S_1+S_2 in configuration space, i.e. the q_1,q_2 position (ignoring other position coordinates not affected by the Stern–Gerlach interaction at S_1), can be associated with just one of the wave packets $\varphi(q_2)\varphi(q_1+gt)$ or $\varphi(q_2)\varphi(q_1-gt)$. So in a subsequent measurement of the spin of S_2 via a Stern–Gerlach interaction at S_2, the evolution of q_2 will depend on the configuration space position of S_1+S_2, which is effectively correlated either with the spin state $|+\rangle_1|-\rangle_2$ or with the spin state $|-\rangle_1|+\rangle_2$. It follows that S_1+S_2 will behave in the S_2-interaction as if its quantum state has effectively collapsed to $\varphi(q_2)\varphi(q_1+gt)|+\rangle_1|-\rangle_2$ or $\varphi(q_2)\varphi(q_1-gt)|-\rangle_1+|\rangle_2$, and so q_2 will evolve to a position corresponding to a z-spin value for S_2 opposite to the z-spin value correlated with the final position of q_1. Evidently, then, the outcome of a spin measurement at S_2 will depend on the type of spin measurement at S_1, i.e. on the orientation of the Stern–Gerlach magnet at S_1. The quantum state of S_1+S_2 will evolve differently for a measurement of x-spin at S_1 rather than z-spin, say, and so will affect the evolution of q_2 differently in a subsequent spin measurement at S_2, as in the measurement of H and H' on the spin-1 system.

There is another sense, though, in which both the functional relationship constraint and the locality condition are satisfied by Bohm's causal interpretation. Consider the spin-1 system again, and the quantum state:

$$\psi(q,t) = \sum_j c_j\varphi(q-gjt)|H=j\rangle$$

at times $t \geq T$ after the interaction. If q is a preferred determinate observable then, *in this state*, we can take H as determinate and also S_z^2 as determinate for the system, i.e. H and S_z^2 inherit determinate status from q and the quantum state, to the extent that the measurement can be regarded as ideal, and the wave functions $\varphi(q-gjt)$ approximate orthogonal eigenfunctions. Of course, the wave function tails will always overlap to some small extent, so even if we

[52] D. Bohm, *Quantum Theory* (Englewood Cliffs, NJ: Prentice-Hall, 1951).

discretize q to three relevant values (representing three non-overlapping ranges of q-values associated with the three peaks of the wave functions), the projections of the state $e(\psi(q,t))$ onto the three corresponding eigenspaces will not yield the rays e_{q_j} spanned by the vectors $\varphi(q-gjt)|H=j\rangle$, for $j = -1,0,+1$, but only rays e'_{q_j} that are arbitrarily close to the rays e_{q_j} (for sufficiently large t). Here we simply treat the wave functions $\varphi(q-gjt)$ as non-overlapping and effectively eigenfunctions of q, as Bohm does in his analysis of measurement in quantum mechanics.[53] Then H-propositions and S_z^s-propositions (more precisely, propositions arbitrarily 'close to' H-propositions and S_z^2-propositions) belong to the lattice $L_{e_{q_0}e_{q_{-1}}e_{q_{+1}}}$. By the theorem of Section 2, $L_{e_{q_0}e_{q_{-1}}e_{q_{+1}}}$ contains the maximal set of propositions that we can take as determinately true or false for the state $\psi(q,t)$, given that q has a determinate value—the maximal set of propositions associated with observables that are measured via the pointer q in the Bohmian sense. For the state $\psi(q,t)$, observables like H and S_z^2 associated with $L_{e_{q_0}e_{q_{-1}}e_{q_{+1}}}$ satisfy the functional relationship constraint. The value assigned to S_z^2 is the square of the value assigned to H, for any position of the pointer q, and similarly for other observables associated with $L_{e_{q_0}e_{q_{-1}}e_{q_{+1}}}$. ($H'$-propositions do not, of course, belong to $L_{e_{q_0}e_{q_{-1}}e_{q_{+1}}}$.) So, if we consider the set of observables that are measured in the Bohmian sense, by the evolution of the quantum state to a specific form that correlates the preferred determinate observable to values of these other observables, then for these observables the functional relationship constraint is satisfied. It follows that the locality condition is satisfied in *this* sense, for composite systems associated with tensor product Hilbert spaces, because locality is a special case of the functional relationship constraint.

Bohm's interpretation is a 'beable' interpretation of quantum mechanics in the spirit of Bell's notion. As Bell put it:[54]

> It would be foolish to expect that the next basic development in theoretical physics will yield an accurate and final theory. But it is interesting to speculate on the possibility that a future theory will not be *intrinsically* ambiguous and approximate. Such a theory could not be fundamentally about 'measurements,' for that would again imply incompleteness of the system and unanalyzed interventions from outside. Rather it should again become possible to say of a system not that such and such may be *observed* to be so but that such and such *be so*. The theory would not be about '*observ*ables' but about '*be*ables'.[...]
>
> Many people must have thought along the following lines. Could one not just promote *some* of the 'observables' of the present quantum theory to the status of beables? The beables would then be represented by linear operators in the state space. The values which they are allowed to *be* would be the eigenvalues of those operators. For the general state the probability of a beable *being* a particular value would be calculated just as was formerly calculated the probability of *observing* that value.

[53]See Chap. 6, 'Measurement as a Special Case of Quantum Process', in D. Bohm and B. J. Hiley, *The Undivided Universe: An Ontological Interpretation of Quantum Theory, op. cit.*, pp. 97–133.

[54]J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics, op. cit.*, p. 41.

Bell's notion of a 'beable,' referring to an objective property of a physical system, is equivalent to Einstein's 'element of reality', the terminology employed in the Einstein–Podolsky–Rosen argument.[55] Bell writes:[56]

> In particular we will exclude the notion of 'observable' in favour of that of '*be*able.' The beables of the theory are those elements which might correspond to elements of reality, to things which exist. Their existence does not depend on 'observation'. Indeed observation and observers must be made out of beables.

The 'causal' interpretation, as Bohm and Hiley characterize it,[57] is a reformulation of quantum mechanics in terms of beables:

> This theory is formulated basically in terms of what Bell has called 'beables' rather than of 'observables.' These beables are assumed to have a reality that is independent of being observed or known in any other way. The observables therefore do not have a fundamental significance in our theory but rather are treated as statistical functions of the beables that are involved in what is currently called a measurement.

The interpretation of quantum mechanics in terms of beables is motivated by certain realist principles formulated by Einstein, a separability principle and a locality principle. These principles are implicit in the Einstein–Podolsky–Rosen argument (which appears to have been largely written by Podolsky) and explicit in various reformulations of the argument by Einstein. For example, referring to the Einstein–Podolsky–Rosen argument, Einstein writes as follows:[58]

> Of the 'orthodox' quantum theoreticians whose position I know, Niels Bohr's seems to me to come nearest to doing justice to the problem. Translated into my own way of putting it, he argues as follows:
>
> If the partial systems A and B form a total system which is described by its ψ-function $\psi/(AB)$, there is no reason why any mutually independent existence (state of reality) should be ascribed to the partial systems A and B viewed separately, *not even if the partial systems are spatially separated from each other at the particular time under consideration.* The assertion that, in this latter case, the real situation of B could not be (directly) influenced by any measurement taken on A is therefore, within the framework of quantum theory, unfounded and (as the paradox shows) unacceptable.
>
> By this way of looking at the matter it becomes evident that the paradox forces us to relinquish one of the following two assertions:
>
> (1) the description by means of the ψ-function is complete
> (2) the real states of spatially separated objects are independent of each other.

[55]A. Einstein, B. Podolsky and N. Rosen, 'Can Quantum–Mechanical Description of Physical Reality be Considered Complete?', *Physical Review* **47** (1935), 777–780.

[56]J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press, 1987), p. 174.

[57]D. Bohm and B. J. Hiley, *The Undivided Universe: An Ontological Interpretation of Quantum Theory op. cit.*, p. 41.

[58]A. Einstein, in *Albert Einstein: Philosopher–Scientist*, P.A. Schilpp (ed.) (Illinois: Open Court, 1991), 3rd edn, pp. 681–682.

On the other hand, it is possible to adhere to (2), if one regards the ψ-function as the description of a (statistical) ensemble of systems (and therefore relinquishes (1)). However, this view blasts the framework of the 'orthodox quantum theory.'

Einstein's 'real state' or 'real situation' of a physical system (elsewhere[59] he speaks of the 'being-thus' of a system) corresponds in our formulation to a 2-valued homomorphism on the determinate sublattice defined by the quantum state of the system and the preferred determinate observable. Each 2-valued homomorphism selects a particular determinate value for the preferred observable, and also particular determinate values for other observables associated with the determinate sublattice, i.e. each 2-valued homomorphism selects a set of preferred and derived determinate properties for the system. Since a determinate sublattice is uniquely defined by the quantum state and a preferred determinate observable, and the 2-valued homomorphisms on a determinate sublattice are in 1–1 correspondence with the values of the preferred observable, Einstein's 'real state' can be characterized equivalently as the specification of the quantum state of the system and the value of the preferred observable, which constitute the only genuine beables of the system (other observables associated with a determinate sublattice represent dispositions in the sense discussed above).

The separability and locality principles can therefore be formulated as follows:

Separability: The determinate properties ('real states') of spatially separated systems are independent of each other.

Locality: If two systems are spatially separated, then the determinate properties ('real state') of one system cannot be directly influenced by any measurement on the other system.

As Einstein presents it, the issue of the completeness of quantum mechanics—the heart of the dispute between Einstein and Bohr—concerns the separability principle. What our determinate sublattices preserve is only a weak separability principle: the determinate properties ('real states') of spatially separated systems are independent of each other, i.e. each system is independently characterized by its own determinate sublattice, if and only if the quantum state of the composite system is *not* an 'entangled' state (linear superposition of product states) arising from past interaction between the systems.

Several authors, notably Fine[60] and Jammer,[61] have suggested that Einstein had something other than hidden variables in mind when he argued that

[59]A. Einstein, *Dialectica* **2** (1948), 320–324. Translated as 'Quantum Mechanics and Reality', in M. Born (ed.), *The Born–Einstein Letters* (London: Walker & Co., 1971), p. 192.

[60]A. Fine, *The Shaky Game: Einstein, Realism, and the Quantum Theory* (Chicago: University of Chicago Press, 1986).

[61]M. Jammer, *The Philosophy of Quantum Mechanics* (New York: Wiley, 1974).

quantum mechanics is incomplete. Einstein's negative reaction to Bohm's hidden variables theory in correspondence with Renninger, Born and others is often cited in support of this view. (In a letter to Born dated 12 May 1952, Einstein dismissed the theory as 'too cheap for me.'[62]) However, we agree with Bell's endorsement[63] of Shimony's characterization of Einstein as 'the most profound advocate of hidden variables', in the sense of a 'beable' interpretation of quantum mechanics. Einstein's lack of enthusiasm for Bohm's theory should not be construed as a rejection of the hidden variables program *per se*, but only a particular way of developing this program. What our theorem shows is that the possible 'completions' of quantum mechanics in Einstein's sense can be uniquely characterized and reduced to the choice of a fixed preferred determinate observable, i.e. a fixed beable. So, in fact, the option of preserving separability in the strong sense is excluded in a beable interpretation.

We have noted that the determinate sublattices satisfy the Kochen and Specker functional relationship constraint in what might be termed an 'ontological' sense, i.e. the values of the determinate observables associated with a determinate sublattice, as assigned by all 2-valued homomorphisms on the lattice, preserve the functional relationships satisfied by these observables, and hence preserve locality as a special case of the functional relationship constraint. But in what might be termed a 'dynamical' sense, the interpretations associated with determinate sublattices are non-local. If we understand a measurement as an interaction that induces an evolution of the quantum state to a form that correlates values of a preferred determinate observable with values of other observables, in virtue of which these other observables achieve derived determinate status in the time-evolved state, then a measurement on a system S_1 can make determinate a dispositional property of a system S_2, spatially separated from S_1, that was not determinate before the measurement. And that will translate into a different value for the preferred observable when the disposition is actualized in a measurement. In this sense, Einstein's locality principle is violated. The 'real states' of one system can be directly influenced by a measurement on another spatially separated system.

3.5. *Bohr's Complementarity Interpretation*

While beable interpretations of quantum mechanics like Bohm's interpretation take a fixed preferred observable as determinate once and for all, on Bohr's complementarity interpretation an observable can be said to have a determinate value only in the context of a specific, classically describable experimental arrangement suitable for measuring the observable. For Bohr, a quantum 'phenomenon' is an individual process that occurs under conditions defined by a specific experimental arrangement. The experimental arrangements

[62]A. Einstein, in M. Born (ed.), *The Born–Einstein Letters* (London: Walker & Co., 1971), p. 192.
[63]J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics, op. cit.*, p. 89.

suitable for locating an atomic object in space and time, and for a determination of momentum–energy values, are mutually exclusive. We can choose to investigate either of these 'complementary' phenomena at the expense of the other, so there is no unique description of the object in terms of determinate properties.

Summing up a discussion on causality and complementarity, Bohr writes:[64]

> Recapitulating, the impossibility of subdividing the individual quantum effects and of separating a behaviour of the objects from their interaction with the measuring instruments serving to define the conditions under which the phenomena appear implies an ambiguity in assigning conventional attributes to atomic objects which calls for a reconsideration of our attitude towards the problem of physical explanation. In this novel situation, even the old question of an ultimate determinacy of natural phenomena has lost its conceptual basis, and it is against this background that the viewpoint of complementarity presents itself as a rational generalization of the very ideal of causality.

Pauli characterizes Bohr's position this way:[65]

> While the means of observation (experimental arrangements and apparatus, records such as spots on photographic plates) have still to be described in the usual 'common language supplemented with the terminology of classical physics', the atomic 'objects' used in the theoretical interpretation of the 'phenomena' cannot any longer be described 'in a unique way by conventional physical attributes'. Those 'ambiguous' objects used in the description of nature have an obviously symbolic character.

We can understand the complementarity interpretation as the proposal to take the classically describable experimental arrangement (suitable for either a space–time or a momentum–energy determination) as defining the preferred determinate observable in what Bohr calls a quantum 'phenomenon'. So the preferred determinate observable is not fixed for a quantum system, but is defined by the classically described 'means of observation'. On this view, the determinate sublattice of a quantum system depends partly on what and how we choose to measure, not on objective features of the system itself. To echo Pauli, the properties we attribute to a quantum object in a measurement are 'ambiguous,' or merely 'symbolic.' The complementarity interpretation, unlike a beable interpretation, which selects a fixed preferred determinate observable, is not a realist interpretation.[66]

It is instructive to consider the application of the complementarity interpretation to the Einstein–Podolsky–Rosen experiment. Referring to this

[64]N. Bohr, *Dialectica* **2** (1948), 312–319.

[65]W. Pauli, *Dialectica* **2** (1948), 307–311.

[66]It could also be argued that the complementarity interpretation does not really solve the measurement problem, unlike the other 'no collapse' interpretations we have been discussing (excluding the orthodox Dirac–von Neumann interpretation), since Bohr gives no measurement-independent prescription for what R should be taken to be. We concentrate on recovering other aspects of this interpretation, once Bohr's choice of R is granted.

experiment, Bohr remarks[67] that the difference between the position coordinates of two particles, $Q_1 - Q_2$, and the sum of their corresponding momentum components, P_1+P_2, are compatible observables, i.e. they are represented by commuting operators. (This follows immediately from the commutation relation for position and momentum, $QP-PQ=i\hbar I$.) So we can prepare a quantum state in which both these observables have determinate values. It follows that a measurement of either Q_1 or P_1 on the first particle will allow the prediction of the outcome of a subsequent measurement of either Q_2 or P_2, respectively, on the second particle. Or putting it another way, the assignment of a determinate value to Q_1 or P_1 will fix a determinate value for Q_2 or P_2, respectively. But now it would appear, if the two particles are separated and no longer interacting, that the second particle must have both a determinate Q_2-value and a determinate P_2-value prior to the Q_1 or P_1 measurements, which contradicts the assumption that the quantum state is a complete description of the system (since no quantum state assigns determinate values to two incompatible observables). What this argument fails to note, says Bohr, is that the experimental arrangements that allow accurate measurements of Q_1 and P_1 are mutually exclusive, so the predictions concerning Q_2 and P_2 refer to complementary phenomena.

It is unclear from Bohr's discussion how the attribution of a determinate value to an observable Q_1 (or P_1) of a system via a measurement on that system can make determinate an observable Q_2 (or P_2) of a second system spatially separated from and not interacting with the first system—how, that is, Q_2 (or P_2) can inherit determinate status from the selection of Q_1 (or P_1) as a preferred determinate observable. This puzzle is resolved if we take Bohr as proposing that the determinate sublattice for the 2-particle system is the determinate sublattice defined by a simultaneous eigenstate of the observables $Q_1 - Q_2$ and P_1+P_2, and the observable Q_1 (or P_1) as the preferred determinate observable (on the basis of the particular experimental arrangement introduced for the measurement on the first particle). For then the determinate sublattice will contain propositions associated with the observable Q_2 (or P_2) of the second particle, in virtue of the form of the quantum state of the composite system as a strictly correlated linear superposition of product states. The non-locality here is analogous to the non-locality of dispositions discussed above for Bohm's causal interpretation.

Thus, the framework for interpretations of quantum mechanics presented here accommodates Bohr's complementarity interpretation as well as Einstein's realism, in the beable sense (stripped of separability/locality requirements, as in Bohm's interpretation). The opposing positions appear as two quite different proposals for selecting the preferred determinate observable—either fixed, once

[67]See Bohr, *op. cit.*, p. 316.

and for all, as the realist would require, or settled pragmatically by what we choose to observe.

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